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Option pricing

- We will now seek to determine the value of options. Unlike futures and swaps,k the payoff of options at maturity is asymmetric in that it will only be exercised if the buyer finds it profitable to do so, imposing a loss on the seller.
- This asymmetry will give rise to the option always have a positive value, but it will also make the value of the underlying asset at maturity a central element for the value of an option.

The difficulty in pricing options

- Option pricing is a major field in financial economics
- Assumptions on the way asset prices evolve into the future are an essential ingredient into models
- Once such a stochastic process has been identified, solving for option vales is not trivial
- In many cases no analytical solutions can be obtained, with numerical methods or Monte-Carlo simulations being employed

The difficulty in pricing options

→ Option pricing is fraught with significant difficulties arising from the asymmetric payoff at maturity, which puts a floor of zero on the profits made using options.

- Option pricing is an important field in finance, in asset pricing in particular, and much research efforts is spent on the pricing of exotic options, but also standard options taking into account many empirical properties of asset prices.
- An essential role plays the way the evolution of future asset prices are modelled as these affect the payoffs at maturity; with futures and swaps whose payoffs are symmetric many aspects of this evolution will can out and can be ignored. The asymmetric payoff of options doe snot allow for that.
- Asset prices are usually modelled as a stochastic process and many such processes have been proposed to obtain realistic properties of asset prices. Once a stochastic process has been identified, this needs to be solved for the option value and in most cases this is difficult, requiring advanced mathematical methods in stochastics and partial differential equations.
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    - The complexity of such stochastic processes and the resulting equations often does not allow for a formula to be derived.
    - Solving for options values can only be done using numerical methods or even Monte-Carlo simulations, making the determination of option values time and resources intensive.

→ Given these difficulties, we will limit ourselves here only to the most basic form of option pricing, which nevertheless will allow basic insights into this field.

Restrictive assumptions

- In order to obtain some insights into option pricing, restrictive assumptions are necessary
- These assumptions will allow to derive explicit solutions, but may not meet the reality of how the price of the underlying asset evolves
- While more realistic assumptions provide a better match with observed prices, the general insights restrictive models provide remain valid

Restrictive assumptions

→ In order to be able to derive the value of options explicitly by using formulae, we will make restrictive assumptions on the way the price of the underlying asset evolves over time.

- Making restrictive assumptions about the behaviour of asset prices is necessary to obtain some easy insights into the properties of option values without having to resort to extensive computer experiments.
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  - When making these assumptions, we can obtain an explicit solution, either through a formula or by working through a small number of steps.
  - However, the strong assumption will neglect some important properties of asset prices and how they might affect option prices; we need to be aware of the limits of such an analysis when comparing our results with results in actual markets.
- Making less restrictive assumptions will give us option values that are closer to actually observed option prices, but the general insights that can be gained from the more restrictive models will not change significantly.

→ Thus, making restrictive assumptions simplifies the analysis substantially without reducing the general insights we can obtain.

Arbitrage pricing

- The basic concept in options pricing is that the payoff profile of an option is matched with that of an portfolio of assets whose values are known
- Analysing the composition of this portfolio will then lead to the value of the option
- The simplest form of assumptions is to assume that asset prices move discretely from one time period to another
- This is commonly known as the binomial model of option pricing

Arbitrage pricing

- → Option pricing is commonly done by arbitrage, that is avoiding that investors can make a profit without taking risk and without making an initial investment.
- ▶ In option pricing, the basic idea is that we assemble a portfolio of assets whose value we know and match the value to that of an option at its maturity.
- ▶ We will then analyse how this portfolio is composed and then obtain the value of the portfolio prior to maturity of the option, giving us the value of the option.
- ▶ We will start by assuming that the asset prices move randomly up or down a fixed amount at discrete pints until maturity of the option.
- ▶ This approach is the binomial model of option pricing.
- → We will present the basic idea of this model here to gain some initial understanding of the option pricing approach via arbitrage.

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Binomial pricing of options

→ The binomial model is simple, but also highly flexible as it can be used to determine the value of any option, using any way the value of the underlying asset may evolve. However, we will restrict ourselves to basic cases.

Benefits and limitations of binomial option pricing

- Binomial option pricing is very flexible in that any form of option can be analysed
- The changes in the price of the underlying asset can also be modelled flexibly by making different assumptions
- Using computers, a large number of time periods can be considered, making this methodology realistic by allowing frequent price changes
- A major limitation of binomial asset pricing is that no analytical solution exists and general properties of options prices can only be analysed numerically

Benefits and limitations of binomial option pricing

→ Knowing how binomial option pricing works, we can now briefly discuss the benefits of this approach.

- While we showed the example of a standard call option when discussing the binomial model, any type of options can be priced.
- We can also change how the value of the underlying asset evolves over time in any way we desire, as long as in every step it can only take two values, although a trinomial models, where asset prices can take three values, also exist.
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  - While the calculation for each step is time-consuming, we can use computers to conduct such calculations fast.
  - This will allow us to model a larger number of small steps prior to the maturity of the option. We will use such an approach to obtain the Black-Scholes option pricing formula.
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  - The absence of an analytical solution, a formula, is the major drawback of the binomial model, despite the ability to use computers to calculate option values.
  - For this reason some properties of option prices can only be analysed numerically, and it can be difficult to provide a comprehensive analysis.

→ We will therefore look at a basic option pricing model that allows us to obtain such analytical results.

Providing an explicit solution for option prices

- Seeking an analytical solution to the value of an option, additional assumptions need be made to allow for an explicit solution
- The Black-Scholes model is the most widely used model for standard European options
- It can be derived using different approaches, but is in all cases involving advanced statistical and mathematical methods

Providing an explicit solution for option prices

→ We will now introduce the Black-Scholes model of option pricing as the most fundamental of such approaches providing us with an analytical solution.

- In order to obtain a formula for the option value, we cannot simply take the binomial model and increase the number of steps until time to maturity; additional assumption will be required to provide such a solution.
- The Black-Scholes model was the first option pricing model provided that gave an explicit formula and it is still widely used, but it is only valid for standard European options.
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    - The Black-Scholes formula has been obtained in at least three different ways.
    - Whichever way is chosen, the derivation required higher level statistics abd/or mathematics.

→ For this reason, we will not explicitly derive the Black-Scholes formula, but only provide the idea that leads to it.

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Black-Scholes model

→ We can now look at the Balck-Scholes formula and we will see how it relates to the binomial model.

Limitations to markets with specific properties

- Assuming that asset values are log-normally distributed allows an analytical formula of the value of an option
- This formula is restricted by the assumptions made and cannot easily be transferred to markets in which these are not fulfilled
- The formula is usually stated for call options only, but using the Put-Call parity, the value of put options can be obtained easily

? Why do investor purchase options and not routinely replicate them themselves?

! In order to ensure the value of the option is met, the Δ and loan amount needs to be maintained at all times, but it changes as parameters change - including time to maturity - and thus requires constant (costly) updating of these holdings

Limitations to markets with specific properties

→ Given that the Black-Scholes model required many restrictive assumptions, the option value will only apply in markets whose properties are sufficiently close to these assumptions.

- A key assumption was that asset prices are log-normally distributed. Without this assumption, the Black-Scholes formula cannot be derived.
- Hence in markets where asset prices are not approximately log-normal, or asset returns normally distributed, the option prices obtained might not be a good approximation. This assumption is most notably not fulfilled in commodity markets and also many foreign exchange markets show substantial deviations for log-normality. The same is also true, though less pronounced, for smaller stocks, most notably in the technology sector.
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  - The Black-Scholes formula is normally only stated for standard Call options.
  - The value of standard Put options can easily be obtained using the Put-Call parity.
- [?] We have seen that options can be replicated using the underlying asset and a loan; so why does anyone buy actually options and not do the replication of options themselves?
- [!] We can replicate the option, but the Δ of the option changes as the underlying asset changes its value, or just by the time to maturity reducing. Thus the portfolio replicating the option needs to be constantly adjusted and this is time consuming to set up and any trading costs associated with adjusting values can be substantial, making options a more efficient tool if available.

→

Knowing properties to use options

- Option pricing formulae involve a large number of parameters that affect their value
- Current price of the underlying asset, strike price, volatility, time to maturity, risk-free rate
- Knowing how these parameters affect option prices also allows investors to completely hedge their positions

Knowing properties to use options

- → The Black-Scholes formula is complex and the relationship between parameters and the option value is not always immediately clear.
- Looking at the Black-Scholes formula, we see that it depends on a large number of parameters
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  - There is the price of the underlying asset S,
  - the price at which the option can be exercised K,
  - the volatility of the asset price, σ,
  - the time to maturity of the option T,
  - and finally the risk-free rate, r
- If we know how these parameters affect the value of options, we will how we can use these properties to ensure that using options risks can be completely eliminated.
- → We will now have a closer look at the effect these parameters have on the option value and how this can be used.

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Properties of option prices

→ We will look at the intuition for the influence of some key parameters and also derive their effect for formally.

Option values and risks

- Options are increasing in the volatility of the underlying asset, making it one of the few asset whose value increases as risks increase
- The increased value can be explained with the increase in utility the insurance against these risks provides

? Why do investors not always use Δ-hedging but instead rely on futures and swaps?

! In order to ensure the value of the position is not changing, the hedge ratio needs to be maintained at all times, but it changes as variables change - including time to maturity - and thus requires constant (costly) updating of option holdings

Option values and risks

→ We see that options vary with as any parameters change, but this dependence on parameters can also be used to provide perfect hedges.

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  - We see that options are increasing in the volatility of the underlying asset.
  - Most assets decrease in value if the volatility increases, but options are one of the few assets whose value increases.
- This is because of the protection against losses they provide against any losses that might arise from such volatility.
- [?] Delta hedging completely eliminates and losses, but also any gains, this is similar to a futures contract or a swap. As options have other uses, why have not crowded out futures in the market?
- [!] The Δ of the option changes as the underlying asset changes its value, or just by the time to maturity reducing. Thus the number of options required needs to be constantly adjusted and this is time consuming to set up and any trading costs associated with adjusting values can be substantial, making futures and swaps a more efficient tool if available.

→ Options are not designed to provide perfect hedges, they are designed to provide protection against losses only, but we see that they can be used differently if other derivatives are not available.

Summary

- Option pricing depends critically on the assumptions about the future evolution of the price of the underlying asset
- Explicit solutions for option prices can only be obtained if these assumptions are very restrictive
- Option pricing suggests that they can be replicated using the underlying asset and investment into a risk-free asst (short position)

Summary

→ We can now summarize the key results we have obtained about option pricing.
- The assumptions we make about the evolution of the value of the underlying asset are critical for the value of options.
- Despite this central importance of the evolution of the asset price, we can obtain explicit solutions only if we make very restrictive and unrealistic assumptions.
- A key aspect of option pricing is that option prices can be replicated using the underlying asset and a loan. This shows that options can be created by investors themselves if no options they require are readily available.

→ The field of option pricing, especially for exotic options and other securities that have option-like features, are importance in financial economics, but also for banks and other investors that issue, trade and use such securities.

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