Spine Changes of Measure and Branching Diffusions

submitted by

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For Mum and Dad

Summary

The main object of study in this thesis is branching Brownian motion, in which each particle moves like a Brownian motion and gives birth to new particles at some rate. In particular we are interested in where particles are located in this model at large times T: so, for a function f up to time T, we want to know how many particles have paths that look like f.

Additive spine martingales are central to the study, and we also investigate some simple general properties of changes of measure related to such martingales.

Acknowledgements

I would like to express my gratitude to my supervisor, Simon Harris, for his help throughout my study at Bath. Without his advice and encouragement, not to mention many good ideas, this thesis would not have been possible.

I also wish to thank the rest of the probability laboratory at Bath, and all the support staff, for providing such a friendly and supportive working environment.

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From many of my teachers at Arnold Hill School, through the lecturers and supervisors at Cambridge — particularly those from Clare College — and all the staff at Bath, I have been lucky to be taught by many superbly knowledgeable and helpful people, and this thesis is the product of their endeavours as much as my own.

I thank all my friends from over the years: those in Nottingham, everyone at Clare, my officemates Marcel Ortgiese and Geoff Littler, the Bath Venturers cricket team, everyone who ever came to the Parade on a Friday, and all the others at Bath for making my time here so enjoyable.

Finally, my family has never wavered in their support over my many years in education, and I thank them all — but especially my parents, who have given me so much.

When I was young, my mother taught me three things:
Beware of the wolves in the forest;
Watch out for ladies of uncertain virtue;
And never, ever *divide by zero.*

Contents

Chapter 1

Introduction

1.1 Branching Brownian motion

Branching Brownian motion is the most fundamental of branching diffusions. The basic model can be described quite simply: fix a real number $r > 0$, known as the *branching rate*, and a random variable A taking values in $\mathbb{N}$ and satisfying $m := \mathbb{E}[A] \in (1, \infty)$, known as the *birth distribution*. Then:

- We begin with one particle at the origin.
- Each particle, during its lifetime, moves according to a Brownian motion in $\mathbb{R}$, independently of all other particles.
- Each particle's lifetime is exponentially distributed with parameter r, independently of its position and of all other particles.
- Each particle dies at the end of its lifetime, leaving in its place a random number of offspring. This random number has the same distribution as the random variable $1 + A$. Relative to their birth time and position, the offspring act independently of each other.

We let $N(t)$ be the set of particles that are alive at time t, and for $u \in N(t)$ we denote its position at time t by $X_u(t)$. We extend the notion of position for $u \in N(t)$ to include the ancestors of u, so if $v \in N(s)$ for some $s < t$ and v is an ancestor of u, then we set $X_u(s) := X_v(s)$.

There are several changes that can be made to this model. We could start from a more general distribution of particles; the diffusion of particles could be in $\mathbb{R}^d$ for any $d \geq 1$; the branching rate r for a particular particle might depend on that particle's position, as might the birth distribution A; and we might allow A to take the value -1, so that particles may die without giving birth to any offspring. Some of these possibilities will be encountered later in this thesis. There is also no reason why we should restrict to Brownian motion: we could easily consider other diffusions, for example, or even many more general Markov processes. We note that many authors specify $1 + A$, rather than A itself, as the birth distribution. We run against this more intuitive convention simply because our choice simplifies notation in later chapters.

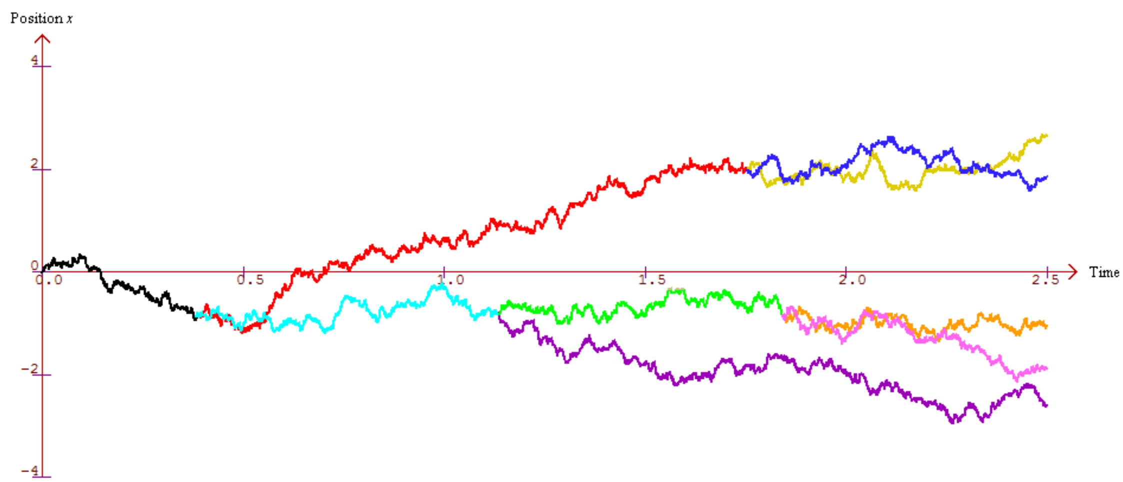

Figure 1-1: Simulation of a branching Brownian motion with a branching rate of 1 and offspring distribution $A \equiv 1$.

Figure 1-1 shows an example of a branching Brownian motion: around time $t = 0.4$, our initial particle (black) dies and gives birth to two new particles (red and cyan) which, given their birth time and position, move as independent Brownian motions. Some time later one of these particles (cyan) dies and gives birth to two new particles (green and purple) so that — at time $t = 1.5$, for example — we have three particles alive. One of these (red) again dies and gives birth to two new particles (blue and yellow), leaving us with four particles, and finally just before time $t = 2$ the green particle dies and two more (pink and orange) are born. Thus at time $t = 2.5$ there are 5 particles alive.

Many interesting questions are immediately apparent. First we may ask how many particles are alive at a fixed time t — but since this does not depend on the position of the particles, it is easily addressed — see for example Haccou *et al.* [9]. We are much more interested in where the particles are located.

1.1.1 The position of the right-most particle

A glance at Figure 1-2 suggests that the particles in a BBM fill out a triangular shape, and that it might be interesting to look at the position of the extremal particle — the particle

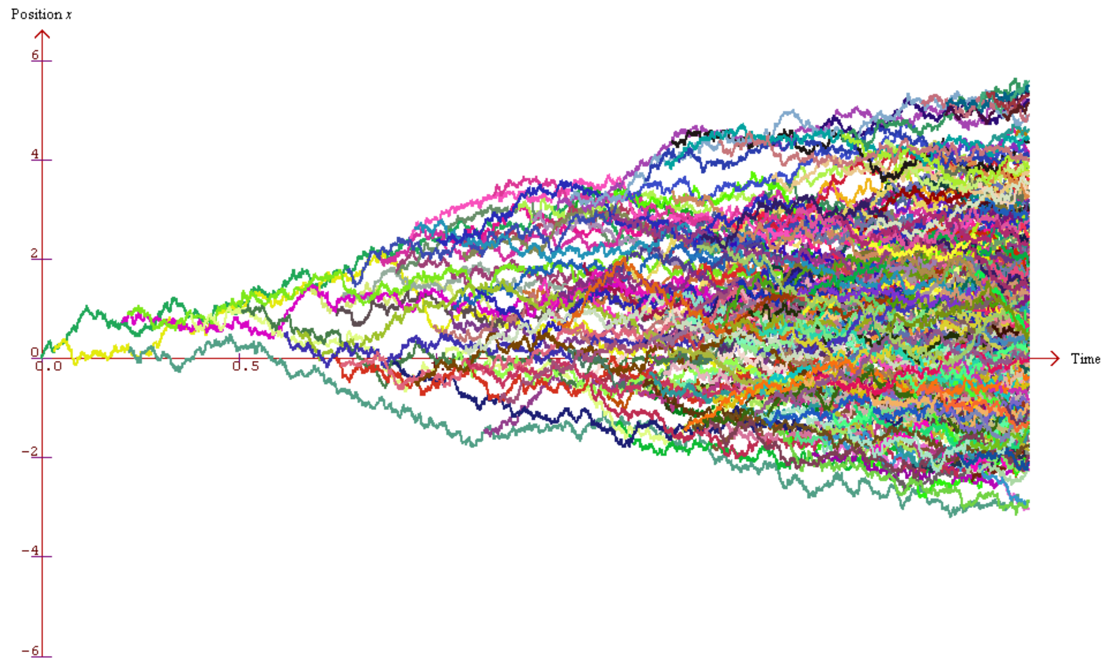

Figure 1-2: Simulation of a branching Brownian motion with a branching rate of 3 and offspring distribution $A \equiv 1$.

with maximal position at some time $t \geq 0$. One might immediately conjecture that its *speed* — its position divided by time — converges to a constant as $t \to \infty$. Indeed this is true, and it is not difficult to prove that the constant is $\sqrt{2r}$. In fact more precise results are available. One such beautiful result was given by Bramson [4] via some powerful and explicit analysis of the Brownian bridge.

Theorem 1.1 (Bramson [4])**:**
For the branching Brownian motion with breeding rate $r = 1$ and branching distribution $A \equiv 1$, and any $\varepsilon \in (0,1)$, $t > 0$,

$$\mathbb{P}\left(\max_{u \in N(t)} X_u(t) \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + b_\varepsilon(t)\right) = \varepsilon$$

where $b_\varepsilon(t) = O(1)$ as $t \to \infty$.

Results on the extremal particle for more complicated models with space-dependent branching were given by Harris and Harris [14]. More recently Hu and Shi [19] have been able to give almost-sure results on the same quantity for a class of branching random walks. One might hope that an almost-sure result with the flavour of Theorem 1.1 holds in the BBM case also. We shall see some results in a similar — but slightly different —

direction in Chapter 5.

1.1.2 BBM with absorption

Once we know the speed of the extremal particle at large times t, we might ask about its history: have its ancestors stayed close to the critical speed throughout, or have they hovered around in the mass of particles near the origin and made a late dash as we get close to time t? One way of interpreting this question is to consider *branching Brownian motion with absorption*. One imagines an absorbing line $\Gamma(t) = -x + \gamma t$ where γ is a constant close to the critical value $\sqrt{2r}$, such that whenever a particle hits the line $\Gamma(t)$ it disappears and is removed from the system. Are there any particles still present at large times? If so then we may consider them to have stayed "close" to the extremal edge of the system.

This model was studied by Kesten [25], who discovered, via some involved estimates on Brownian motion, asymptotics for extinction probabilities and numbers of particles in intervals of the area above the absorbing line. To choose two examples of particular interest, Kesten shows that if $\gamma < \sqrt{2r}$ then there is strictly positive probability that $N(t)$ never becomes empty; and that in the critical case $\gamma = \sqrt{2r}$, the probability that there is at least one particle present at time t is approximately $\exp(-kt^{1/3})$ for some positive constant k. Further results on BBM with absorption, and applications to the Fisher-Kolmogorov-Petrovski-Piscounov (FKPP) equation from mathematical biology, were given by Harris, Harris and Kyprianou [15] and Harris and Harris [13] using more intuitive methods, similar to those used later in this thesis.

1.1.3 Growth along paths

Kesten's results on BBM with absorption tell us that it is possible for particles to stay above the line $\Gamma(t) = -x + \gamma t$ for all time whenever $\gamma < \sqrt{2r}$, and that this is not the case when $\gamma = \sqrt{2r}$. Thus our next question might be: can particles stay within t^β (plus a constant, say) of the critical line for $\beta \in (0,1)$? And if we can answer this question then we might attempt to generalise by moving away from the critical line — given a path $f : [0,\infty) \to \mathbb{R}$, are there particles that stay close to f? How close? This question, phrased more precisely in various ways, becomes the central theme of this thesis.

One interpretation of our question falls in line with the classical large deviations theory for Brownian motion (Schilder's theorem: see Schilder [35] for the original article or Varadhan [36] or Dembo and Zeitouni [6] for more accessible modern formulations) whereby paths on $[0,1]$ are rescaled onto $[0,T]$. This is the approach taken by Git [8], Lee [30] and Hardy and Harris [10], and we follow the same route in Chapter 4. For the purposes of illustration we paraphrase a theorem of Git [8].

Theorem 1.2 (Git [8])**:**
Let $D \subseteq C[0,1]$, and let $m = \mathbb{E}[A]$ be the mean of the offspring distribution. Assume that $\mathbb{E}[A \log A] < \infty$. Define

$$N_T(D,\theta) := \{u \in N(t) : \exists f \in D \text{ with } Tf(t/T) = X_u(t) \;\; \forall t \in [0,\theta T]\}.$$

Let

$$\theta_0(f) := \inf\left\{\theta \in [0,1] : rm\theta - \frac{1}{2}\int_0^\theta f'(s)^2 ds < 0\right\} \in [0,1] \cup \{\infty\}$$

and

$$K(f,\theta) := \begin{cases} rm\theta - \frac{1}{2}\int_0^\theta f'(s)^2 ds & \text{if } \theta \leq \theta_0(f) \\ -\infty & \text{otherwise.} \end{cases}$$

For any closed set $D \subseteq C[0,1]$ and $\theta \in [0,1]$,

$$\limsup_{T\to\infty} \frac{1}{T} \log |N_T(D,\theta)| \leq \sup_{f\in D} K(f,\theta)$$

almost surely, and for any open set $A \subseteq C[0,1]$ and $\theta \in [0,1]$,

$$\liminf_{T\to\infty} \frac{1}{T} \log |N_T(A,\theta)| \geq \sup_{f\in A} K(f,\theta)$$

almost surely.

In Chapter 4 we will see that there is a mistake in the proof of this theorem in [8], and we shall provide an alternative proof. The methods used turn out to be so robust that, with various technical upgrades, we are in fact able to prove an analogous theorem for a more general setup in which particles may breed at a rate which depends upon their position.

1.1.4 Behaviour along unscaled paths

The second interpretation of our main question is more direct, and similar in direction to the work of Novikov [34] in the case of a single Brownian motion. For any two continuous functions $f : [0,\infty) \to \mathbb{R}$ and $L : [0,\infty) \to (0,\infty)$, we may ask how many particles have paths that stay within distance $L(t)$ of the path $f(t)$ for all times $t \geq 0$. That is, let

$$\hat{N}(t) := \{u \in N(t) : |X_u(s) - f(s)| < L(s) \;\; \forall s \in [0,t]\};$$

for which f and L might $\hat{N}(t)$ remain non-empty for all t? How large is $\hat{N}(t)$ in such cases? If $\hat{N}(t)$ is almost surely empty eventually, then at what rate does the probability $\mathbb{P}(\hat{N}(t) \neq \emptyset)$ decay? These questions are considered in Chapter 5. In order to state the main theorem of that chapter, let

$$S := \liminf_{t\to\infty} \frac{1}{t} \int_0^t \left(r - \frac{1}{2} f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)} \right) ds$$

(if this exists), and

$$\Upsilon := \inf\{t \geq 0 : \hat{N}(t) = \emptyset\}.$$

We assume that certain conditions on f and L hold but we omit those here; we shall examine them in detail in Chapter 5. We also assume that $A \equiv 1$, so we have only binary branching.

Theorem 1.3:
If $S < 0$, then $\Upsilon < \infty$ almost surely and

$$\frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{\inf_{s\leq t} \int_0^s \left(r - \frac{1}{2} f'(u)^2 - \frac{\pi^2}{8L(u)^2} + \frac{L'(u)}{2L(u)} \right) du} \longrightarrow 1.$$

On the other hand, if $S > 0$, then $\mathbb{P}(\Upsilon = \infty) > 0$ and almost surely on the event $\{\Upsilon = \infty\}$ we have

$$\frac{\log |\hat{N}(t)|}{\int_0^t \left(r - \frac{1}{2} f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)} \right) ds} \longrightarrow 1.$$

This theorem on its own is not quite strong enough to tell us everything we were interested in, particularly questions about whether there are particles staying within t^β of the critical line $\sqrt{2r}t$ (since in this case $S = 0$). However, as a byproduct of the proof we are able to state further, more precise theorems in special cases. Indeed we show that it is possible for particles to stay within t^β of the critical line if $\beta > 1/3$, and not if $\beta < 1/3$ (see Theorem 5.27); if $\beta = 1/3$ then the situation is more complicated (see Theorem 5.28) but it is possible for particles to stay within $t^{1/3}$ of critical if the breeding rate r satisfies $r > \frac{1}{2}\left(\frac{81\pi}{4}\right)^2$.

1.2 Spine changes of measure

One of the many useful properties of Brownian motion is the tremendous number of martingales that can be built around it, and similarly there are many martingales involving branching Brownian motion. In order to give a classical example, we recall that we defined $N(t)$ to be the set of particles alive at time t, and for $u \in N(t)$ we denoted the position of u at time t by $X_u(t)$. We work under a probability measure $\mathbb{P}$ and let $(\mathcal{F}_t, t \geq 0)$ be the natural filtration of the process. Then for each $\lambda \in \mathbb{R}$ the process

$$Z_\lambda(t) := \sum_{u \in N(t)} e^{-rmt+\lambda X_u(t)-\frac{1}{2}\lambda^2 t}$$

is a non-negative martingale.

Additive martingales such as this one will play a defining role in this thesis. To see why, we change measure by setting

$$\left.\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_\lambda(t). \tag{$*$}$$

The pathwise construction of this measure change was first seen in 1988 in a paper by Chauvin and Rouault [5]:

Theorem 1.4:
Under $\mathbb{Q}_\lambda$ the process can be constructed as follows:

- *starting from the origin, the original particle moves as a Brownian motion with drift λ;*
- *the original particle undergoes fission at an accelerated rate $(1+m)r$, to be replaced by $1 + A^\star$ particles where $\mathbb{Q}_\lambda(A^\star = k) = \frac{(1+k)\mathbb{P}(A=k)}{1+m}$;*
- *one of these particles is chosen uniformly at random to repeat the behaviour of its parent;*
- *each of the remaining particles initiates, relative to its birth position, an independent copy of a $\mathbb{P}$-branching Brownian motion.*

This theorem is stated as a self-standing result in [5], and is not used to prove any of the other results of the paper. However, the identification of the one special chosen line of descent, or *spine*, and the observation that the rest of the process behaves (conditionally given the spine) as under $\mathbb{P}$, were crucial in the development of the subject. It is interesting to note that under $\mathbb{Q}_0$, the motion of particles is not changed at all, but that one particle (the spine) has more children — this corresponds (intuitively at least — it is easier to calculate directly in the Galton-Watson scenario, see [32]) to the fact that if we choose a particle uniformly from all those alive at time t then we expect that it will have had significantly more than rmt children.

The next major contribution on spine changes of measure was not until 1995, when Lyons, Pemantle and Peres [32] gave elegant probabilistic proofs of three of the most classical theorems concerning Galton-Watson trees — the Kesten-Stigum theorem describing the rate of growth for supercritical processes and corresponding results describing the rate of decay of survival probabilities for critical and subcritical processes. Here the discrete setting allowed simplified notation, but the power of the approach was first realised by the use of a spine decomposition, bounding the growth of the process under the changed measure via calculations depending only the spine.

There have been several more contributions to the subject, not least from Kyprianou [28] on branching diffusions, Bertoin and Rouault [2] on homogeneous fragmentations, Athreya [1] on Markov chains, Lyons [31], Biggins and Kyprianou [3] and Hu and Shi [19] for branching random walks, and Engländer and Kyprianou [7] for superprocesses.

The final development that is of major interest to us for the purposes of our results was provided by Hardy and Harris [11]. The authors noted that the space on which our BBM is constructed can be embellished so that, on this richer space, the "spine" identified by Chauvin and Rouault can be seen directly and forms part of the process. The original BBM process can still be seen via the projection onto its natural filtration, and indeed the use of various different filtrations forms an important part of the construction. This allows all measure changes to be carried out via unit mean martingales, so that all measures are probability measures, and offers a significant advantage over previous methods. We shall use much of the same notation as in [11]. This will be developed fully in Chapter 2.

For the purposes of this introduction we simplify matters by omitting many of the details and attempting to paint the picture without worrying about being too rigorous. Suffice to say that the measure change in $(*)$ can be extended to our embellished space to give a new measure $\tilde{\mathbb{Q}}_\lambda$ under which one line of descent is marked out as the spine and behaves as the special particle in the construction of Chauvin and Rouault described above. We simplify things further by considering, for now, only the binary branching case $A \equiv 1$ (so that $m = 1$ and $A^\star \equiv 1$ also).

We aim to prove that the extremal particle in a BBM travels at speed $\sqrt{2r}$, using spine methods. We begin with a simple spine decomposition, which we attempt to explain without proof. We will see a more general version, with proof, in Chapter 2. The σ-algebra $\tilde{\mathcal{G}}_\infty$ in the decomposition is that which sees only the spine (for all times $t \geq 0$) — its position and its genealogy, and nothing about any other particles.

Theorem 1.5 (Simplified spine decomposition)**:**
$\tilde{\mathbb{Q}}_\lambda$-almost surely,

$$\tilde{\mathbb{Q}}_\lambda[Z_\lambda(t)|\tilde{\mathcal{G}}_\infty] = \sum_{u<\xi_t} e^{-rS_u+\lambda\xi_{S_u}-\lambda^2 S_u/2} + e^{-rt+\lambda\xi_t-\lambda^2 t/2}$$

where ξ_t is the position of the spine at time t, and S_u is the death time of particle u.

We see that this theorem tells us about the additive martingale Z_λ, which depends on many particles, in terms of just one particle, the spine. Figure 1-3 shows the idea behind the theorem: given information about the spine (the red line shows its path under $\tilde{\mathbb{Q}}$), at each birth event (represented by a purple dot) along its path a new non-spine particle is born and goes on to draw out a line of descent of its own. This process is distributed — relative to its birth time and position — as a copy of our original $\mathbb{P}$-branching Brownian motion. Therefore each such new particle u, conditional on its birth time and position, contributes to $Z_\lambda(t)$ a martingale term $Z_\lambda^u(t)$ of its own. When this term is projected back onto $\tilde{\mathcal{G}}_\infty$, the optional stopping theorem tells us that its expected contribution is simply its initial value — that is, its value at the time it split from the spine. This is exactly what we see in the sum part of the spine decomposition. The final term in the decomposition is the contribution made to $Z_\lambda(t)$ by the spine itself.

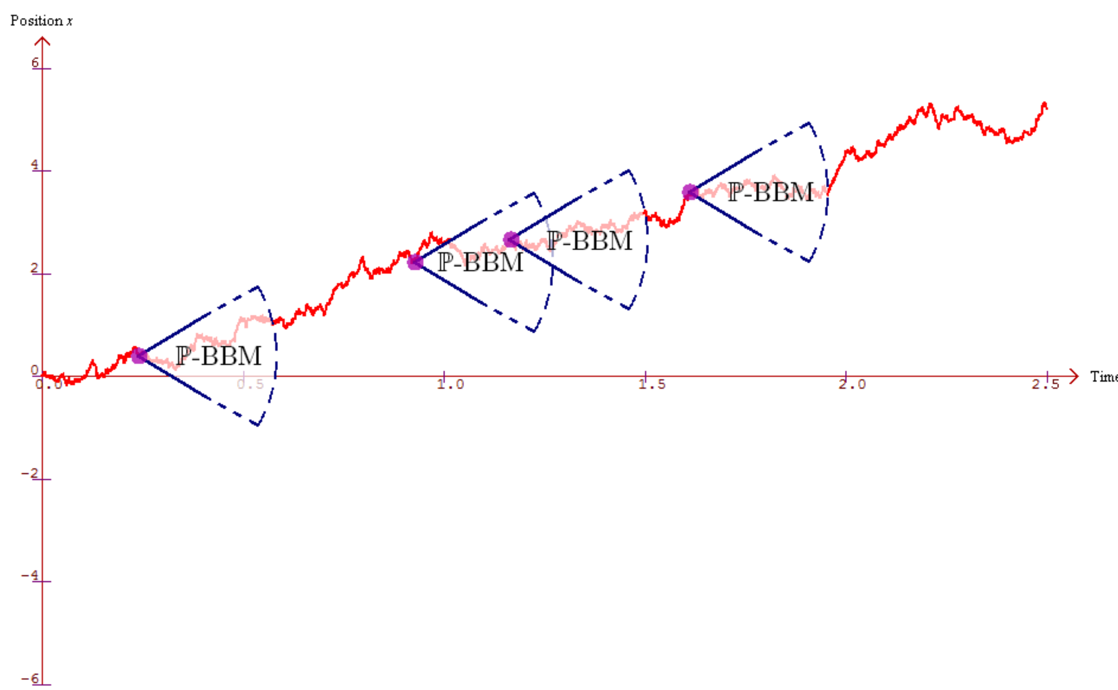

Figure 1-3: The idea behind the spine decomposition.

The following well-known lemma often proves useful in conjunction with the spine decomposition. Again, we shall see the lemma later with a proof; for now we simply state it in order to proceed with an example.

Lemma 1.6:
For any $A \in \mathcal{F}_\infty$ (where $\mathcal{F}_t, t \geq 0$ is the natural filtration of the BBM — without the spine) we have

$$\mathbb{Q}(A) = \mathbb{P}[Z(\infty)\mathbb{1}_A] + \mathbb{Q}(A \cap \{Z(\infty) = \infty\}).$$

We are now in a position to prove the following theorem. As we mentioned in Section 1.1, this result is well-known, but it provides a worthwhile illustration of the simplicity of the spine methods.

Theorem 1.7:
The extremal particle has asymptotic speed $\sqrt{2r}$: that is

$$\frac{\sup_{u\in N(t)} X_u(t)}{t} \to \sqrt{2r} \;\; \mathbb{P}\text{-almost surely.}$$

Proof. First fix $\lambda > \sqrt{2r}$. Suppose that

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} > \lambda\right) > 0.$$

Then there is strictly positive $\mathbb{P}$-probability that there exist particles $u_1, u_2, \ldots$ and times $T_1, T_2, \ldots \to \infty$ such that for each j, $u_j \in N(T_j)$ and $X_{u_j}(T_j) \geq \lambda T_j$. On this event,

$$Z_\lambda(T_j) \geq e^{-rT_j + \lambda X_{u_j}(T_j) - \frac{1}{2}\lambda^2 T_j} \geq e^{-rT_j + \frac{1}{2}\lambda^2 T_j},$$

and hence we see that

$$\mathbb{P}(Z_\lambda(\infty) = \infty) > 0.$$

But $Z(t)$ is a $\mathbb{P}$-martingale, so converges almost surely to an almost surely finite limit, which gives us a contradiction. Thus

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} > \sqrt{2r}\right) \leq \sum_{n=1}^{\infty} \mathbb{P}\left(\limsup_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} > \sqrt{2r} + \frac{1}{n}\right) = 0,$$

establishing our upper bound on the speed of the extremal particle.

For the lower bound we use the spine decomposition. Choose $\lambda > 0$ such that $\lambda < \sqrt{2r}$, fix $p \in (1, \frac{2r}{\lambda^2} \wedge 2)$ and let $q := p - 1$. Using Jensen's inequality and the fact (which is easy to prove: reduce to the case $a = 1$ and differentiate with respect to b) that for any $a, b > 0$ and $q \in (0, 1]$, we have $(a + b)^q \leq a^q + b^q$,

$$\begin{aligned}
\tilde{\mathbb{Q}}_\lambda[Z_\lambda(t)^q | \tilde{\mathcal{G}}_\infty] &\leq \left(\tilde{\mathbb{Q}}_\lambda[Z_\lambda(t) | \tilde{\mathcal{G}}_\infty]\right)^q \\
&= \left(\sum_{u<\xi_t} e^{-rS_u + \lambda\xi_{S_u} - \lambda^2 S_u/2} + e^{-rt + \lambda\xi_t - \lambda^2 t/2}\right)^q \\
&\leq \sum_{u<\xi_t} e^{-qrS_u + q\lambda\xi_{S_u} - q\lambda^2 S_u/2} + e^{-qrt + q\lambda\xi_t - q\lambda^2 t/2}.
\end{aligned}$$

We now take the $\tilde{\mathbb{Q}}_\lambda$-expectation of this quantity. Since, under $\tilde{\mathbb{Q}}_\lambda$, the births along the spine occur as a Poisson process of constant rate $2r$ independently of the position of the spine, standard calculations reveal that

$$\tilde{\mathbb{Q}}_\lambda \left[\sum_{u<\xi_t} e^{-qrS_u + q\lambda\xi_{S_u} - q\lambda^2 S_u/2} \right] = \tilde{\mathbb{Q}}_\lambda \left[\int_0^t 2re^{-qrs+q\lambda\xi_s-q\lambda^2 s/2} ds \right]$$

and by Fubini's theorem we have

$$\tilde{\mathbb{Q}}_\lambda[Z_\lambda(t)^q] \leq \int_0^t 2r\tilde{\mathbb{Q}}_\lambda \left[e^{-qrs+q\lambda\xi_s-q\lambda^2 s/2} \right] ds + \tilde{\mathbb{Q}}_\lambda[e^{-qrt+q\lambda\xi_t-q\lambda^2 t/2}].$$

Since the spine is a Brownian motion with drift λ under $\tilde{\mathbb{Q}}_\lambda$, we may apply Girsanov's theorem to see that for any $s \geq 0$,

$$\begin{aligned}
\tilde{\mathbb{Q}}_\lambda[e^{q\lambda\xi_s-qrs-q\lambda^2 s/2}] &= \tilde{\mathbb{P}}[e^{q\lambda\xi_s-qrs-q\lambda^2 s/2+\lambda\xi_t-\lambda^2 t/2}] \\
&= \tilde{\mathbb{P}}[e^{p\lambda\xi_t-qrt-p\lambda^2 t/2}] \\
&= e^{p^2\lambda^2 t/2-qrt-p\lambda^2 t/2} \\
&= e^{p(p-1)\lambda^2 t/2-(p-1)rt}.
\end{aligned}$$

This exponent is negative by our choice of p, and hence $\tilde{\mathbb{Q}}_\lambda[Z_\lambda(t)^q]$ is bounded over all $t \geq 0$; but

$$\mathbb{P}[Z_\lambda(t)^p] = \mathbb{Q}_\lambda[Z_\lambda(t)^q] = \tilde{\mathbb{Q}}_\lambda[Z_\lambda(t)^q]$$

so $Z_\lambda(t)$ is bounded in $L^p(\mathbb{P})$. By the martingale convergence theorem, $\mathbb{P}[Z(\infty)] = 1$, so by Lemma 1.6 we have

$$\mathbb{Q}(Z(\infty) = \infty) = 1 - \mathbb{P}[Z(\infty)] = 0$$

and for any $A \in \mathcal{F}_\infty$

$$\mathbb{Q}(A) = \mathbb{P}[Z(\infty)\mathbb{1}_A].$$

Thus (using that the spine has drift λ under $\mathbb{Q}$)

$$\mathbb{P}[Z(\infty)\mathbb{1}_{\{\liminf(\sup_{u\in N(t)} X_u(t)/t)<\lambda\}}] = \mathbb{Q}\left(\liminf_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \lambda\right) = 0.$$

We deduce that we must have

$$\mathbb{P}\left(\liminf_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \lambda\right) < 1.$$

But for any $s > 0$, using the fact that given $\mathcal{F}_s$ each particle alive at time s draws out its own independent BBM from time s onward,

$$\begin{aligned}
P(s) &:= \mathbb{P}\left(\liminf_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \lambda \middle| \mathcal{F}_s\right) \\
&= \mathbb{P}\left(\bigcap_{u\in N(s)} \left\{\liminf_{t\to\infty} \frac{\sup_{v\in N(t), u<v} X_v(t)}{t} < \lambda\right\} \middle| \mathcal{F}_s\right) \\
&= \prod_{u\in N(s)} \mathbb{P}_{X_u(s)}\left(\liminf_{t\to\infty} \frac{\sup_{v\in N(t)} X_v(t)}{t} < \lambda\right) \\
&= \mathbb{P}\left(\liminf_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \lambda\right)^{|N(s)|}
\end{aligned}$$

where $\mathbb{P}_x$ denotes a copy of our BBM measure started with a single particle at the point x (rather than at 0). Since $|N(s)| \to \infty$ $\mathbb{P}$-almost surely as $s \to \infty$ (this is a standard result and easy to prove — or see for example [9]), $P(s)$ converges to zero almost surely. But clearly $P(s)$ is a bounded martingale, and hence by the martingale convergence theorem

$$\mathbb{P}\left(\liminf_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \lambda\right) = \mathbb{P}[P(0)] = \mathbb{P}[P(\infty)] = 0.$$

This holds for any $\lambda < \sqrt{2r}$, and so finally

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \sqrt{2r}\right) \leq \sum_{n=1}^{\infty} \mathbb{P}\left(\limsup_{t\to\infty} \frac{\sup_{u\in N(t)} X_u(t)}{t} < \sqrt{2r} - \frac{1}{n}\right) = 0,$$

giving us our lower bound. □

The proof above provides a microcosm of most of this thesis: we want to check the positions of some of the particles in our system; we carefully choose a set of relevant martingales; we use a set of spine tools to bound the growth of the system by changing measure; and to complete the proof we use some kind of "0/1" law to check that our claim holds with probability one.

1.3 Measure changes with extinction

The two sections above suggest that we shall be very interested in investigating "extinction" events using changes of measure. This has been an area of much activity since the original spine papers of Lyons *et al.* [27, 31, 32], as the techniques developed have made possible intuitive proofs of many results both new and old. However the spine methods usually give results conditional on the event that the spine martingale in use has a strictly positive limit, when often one would like results conditional on the event of "survival" of a certain process, usually interpreted to mean the event that the martingale is never zero. Since these two events may be different, we would like to find some reliable way of checking whether they agree (up to a null set). In Chapter 3 we provide an extremely simple necessary and sufficient condition for the two events to agree. We also show that, if $Z(t)$ is a unit mean martingale and we change measure via $\mathbb{Q}|_{\mathcal{F}_t} := Z(t)\mathbb{P}|_{\mathcal{F}_t}$, then

$$\mathbb{P}(Z(t) > 0) = \mathbb{Q}\left[\frac{1}{Z(t)}\right]$$

which corrects a mistake of Lyons [31] and provides another tool for investigating extinction probabilities. We find this useful in Chapter 5.

Chapter 2

The Hardy and Harris spine construction

We aim here to give an overview of the theory behind the spine technique. We take, more or less, the route laid out by Hardy and Harris [11], carrying out all changes of measure by unit-mean martingales to ensure that we work only with probability measures. Many more general results, which are not necessary for our study, may be found in [11].

2.1 The underlying space

2.1.1 Trees

We use the *Ulam-Harris labelling system*: define a set of labels

$$\Omega := \{\emptyset\} \cup \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$$

(as usual $\mathbb{N} = \{1, 2, 3, \ldots\}$).

We often call the elements of Ω *particles*. We think of $\emptyset$ as our "inital ancestor", and a label $(3, 2, 7)$ (for example) as representing "the seventh child of the second child of the third child of the initial ancestor". For a particle $u \in \Omega$ we define $|u|$, the generation of u, to be the length of u (so if $u \in \mathbb{N}^n$ then $|u| = n$, and $|\emptyset| = 0$). For two labels $u, v \in \Omega$ we write uv for the concatenation of u and v, so for example $(3, 2, 7)(1, 5, 4) := (3, 2, 7, 1, 5, 4)$ (and we take $\emptyset u = u\emptyset = u$). We write $u \leq v$ and say that u is an *ancestor* of v if there exists $w \in \Omega$ such that $uw = v$.

We define a *tree* to be a subset $\tau \subseteq \Omega$ such that

- $\emptyset \in \tau$: the initial ancestor is part of τ;

- for all $u, v \in \Omega$, $uv \in \tau \Rightarrow u \in \tau$: if τ contains a particle then it contains all the ancestors of that particle;
- for each $u \in \tau$, there exists $A_u \in \{-1, 1, 2, 3, 4, \ldots\}$ such that for $j \in \mathbb{N}$, $uj \in \tau$ if and only if $1 \leq j \leq 1 + A_u$.

We let $\mathbb{T}$ be the set of all such trees.

2.1.2 Marked trees and branching Brownian motion

Since we wish to have a particular view of trees, as systems evolving in time and space, we define a *marked tree* to be a set τ of triples of the form (u, σ_u, X_u) such that $u \in \Omega$, the set

$$\mathrm{tree}(\tau) := \{u : \exists \sigma_u, X_u \text{ such that } (u, \sigma_u, X_u) \in \tau\}$$

forms a tree, $\sigma_u \in [0, \infty)$ is the *lifetime* of u, and, setting $S_u := \sum_{v \leq u} \sigma_v$,

$$X_u : [S_u - \sigma_u, S_u) \to \mathbb{R}$$

is the *position function* of u. We think of $X_u(t)$ as describing the spatial position of the particle u at time t. To paint the picture more clearly, we think of the inital ancestor $\emptyset$ moving around in space according to its position function $X_\emptyset$ until just before the time $\sigma_\emptyset$. At this time a number $A_\emptyset$ of new particles appear and each moves around in space according to its position function for a period of time equal to its lifetime, before being replaced by a number of new particles; and so on.

We let $\mathcal{T}$ be the set of all marked trees, and for $\tau \in \mathcal{T}$ we define the set of particles alive at time t to be

$$N(t) := \{u \in \mathrm{tree}(\tau) : S_u - \sigma_u \leq t < S_u\}.$$

For convenience, we extend the position path of a particle to all times $t \in [0, S_v)$, to include the paths of all its ancestors:

$$X_v(t) := \begin{cases} X_v(t) & \text{if } S_v - \sigma_v \leq t < S_v \\ X_u(t) & \text{if } u < v \text{ and } S_u - \sigma_u \leq t < S_u. \end{cases}$$

We now construct a probability measure P on $\mathcal{T}$ such that under P, the system evolves as a branching Brownian motion.

Lemma 2.1:
For any $r > 0$ and random variable A taking values in $\mathbb{N}$ with finite mean, there exists

a σ-algebra $\mathcal{H}$ and a unique probability measure P on the space of marked trees under which:

- *The initial particle $\emptyset$ begins at the origin, $X_\emptyset(0) = 0$.*
- *Each particle's lifetime σ_u is exponentially distributed with parameter r, independent of its position and of all other particles.*
- *For each particle u, $(X_u(t) - X_u(S_u - \sigma_u), t \in [0, \sigma_u))$ is a standard Brownian motion started from 0 and independent of all other particles.*
- *For each particle u, the number $1 + A_u$ of offspring of u has the same distribution as the random variable $1 + A$ and is independent of the particle's position and of all other particles.*

Proof. Ikeda *et al.* [20, 21, 22] prove that such a measure exists on some space; by taking the image of that measure on the space of marked trees, we obtain our measure P. It is easily checked that the distribution specified in the lemma is enough to ensure uniqueness over finite time intervals, and hence over the whole space. $\square$

2.1.3 Marked trees with spines

We now enlarge our state space further to include the notion of *spines*, which will be a central theme of this thesis and will allow us certain probabilistic tools without which our study would be significantly more difficult. We define a *spine* to be a single maximal distinguished line of descent. That is, a spine ξ on a marked tree τ is a subset of $\text{tree}(\tau)$ such that

- $\emptyset \in \xi$;
- $\xi \cap N(t)$ contains exactly one particle for each t;
- if $v \in \xi$ and $u < v$ then $u \in \xi$.

If $v \in \xi \cap N(t)$ then we define $\xi_t := X_v(t)$, the position of the spine at time t. At certain points we shall also use the notation ξ_t to mean the particle v itself — beyond this introduction it should always be clear from the context which meaning is intended, and so this should not lead to any ambiguity. For clarity within this section we will use the less concise notation $\text{node}(\xi_t)$ to denote the particle v itself — that is, the unique $v \in N(t) \cap \xi$. We let $\tilde{\mathcal{T}}$ be the set of all marked trees with spines.

2.2 Filtrations

We now work exclusively on the space $\tilde{\mathcal{T}}$ of marked trees with spines, and use four different filtrations on this space, $\mathcal{F}_t$, $\tilde{\mathcal{F}}_t$, $\mathcal{G}_t$ and $\tilde{\mathcal{G}}_t$, to encapsulate different amounts of information.

The filtration $(\mathcal{F}_t, t \geq 0)$
We define $(\mathcal{F}_t, t \geq 0)$ to be the natural filtration of a BBM on $\tilde{\mathcal{T}}$: so if $(\mathcal{H}_t, t \geq 0)$ is the natural filtration of the BBM process defined in Lemma 2.1, then

$$\mathcal{F}_t = \{\{(\tau, \xi) : \tau \in B, \xi \text{ is a spine on } \tau\} : B \in \mathcal{H}_t\}.$$

$\mathcal{F}_t$ contains the all the information about the marked tree up to time t — so, everything about those particles that have lived and died before time t, along with the information up to time t of those particles still alive at time t. However, it does not know which particle is the spine at any point.

The filtration $(\tilde{\mathcal{F}}_t, t \geq 0)$
For each $t \geq 0$ we define a σ-algebra $\tilde{\mathcal{F}}_t$ on $\tilde{\mathcal{T}}$ by

$$\tilde{\mathcal{F}}_t := \sigma\left(\mathcal{F}_t \cup \{\{u = \text{node}(\xi_s)\} : u \in \Omega, s \in [0, t]\}\right)$$

(recall that Ω was the set of all Ulam-Harris labels). $\tilde{\mathcal{F}}_t$ contains all the information about both the marked tree and the spine up to time t.

The filtration $(\mathcal{G}_t, t \geq 0)$
We define

$$\mathcal{G}_t := \sigma\left(\xi_s, s \in [0, t]\right)$$

where ξ_s represents the position of the spine at time s. $\mathcal{G}_t$ contains just the spatial information about the spine up to time t, but does not know which *nodes* of the tree actually make up the spine. It is a Brownian filtration.

The filtration $(\tilde{\mathcal{G}}_t, t \geq 0)$
We define

$$\tilde{\mathcal{G}}_t := \sigma\left(\mathcal{G}_t \cup \mathcal{A}_t \cup \mathcal{C}_t\right).$$

where

$$\mathcal{A}_t = \{\{u = \text{node}(\xi_s)\} : u \in \Omega, s \in [0, t]\}$$

and

$$\mathcal{C}_t = \{\{u < \text{node}(\xi_t), A_u = k, \sigma_u \leq \sigma\} : u \in \Omega, k \in \mathbb{N}, \sigma \in [0, \infty)\}.$$

$\tilde{\mathcal{G}}_t$ contains all the information about the spine up to time t: which nodes make up the spine, their positions, and for all spine nodes not in $N(t)$ (so all the strict ancestors of the spine at time t) their lifetimes and number of children.

We note that $\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t$ and $\mathcal{G}_t \subseteq \tilde{\mathcal{G}}_t \subseteq \tilde{\mathcal{F}}_t$.

2.3 Probability measures

We may define a canonical measure $\mathbb{P}$ on $(\tilde{\mathcal{T}}, \mathcal{F}_\infty)$ as the image of the measure P given by Lemma 2.1: for any $B \in \mathcal{H}$, set

$$\mathbb{P}(\{(\tau, \xi) : \tau \in B, \xi \text{ is a spine on } \tau\}) = P(B).$$

This measure, however, has no knowledge of the spine (since it sees only the filtration $\mathcal{F}_t$). We would like to extend this to a measure on the finer filtration $\tilde{\mathcal{F}}_t$. To do this, we imagine the spine, at each fission event, choosing uniformly from the available children. Then it is easy to see that, for any particle u in a marked tree τ, we would like

$$\text{Prob}(u \in \xi) = \prod_{v<u} \frac{1}{A_v + 1}.$$

We note also that if Y is an $\tilde{\mathcal{F}}_t$-measurable random variable then we can write:

$$Y = \sum_{u \in N(t)} Y_u \mathbb{1}_{\{\xi_t = u\}} \tag{2.1}$$

where each Y_u is $\mathcal{F}_t$-measurable. The proof of this fact is fairly simple. One shows first by direct construction that if

$$A \in \mathcal{F}_t \cup \{\{u = \text{node}(\xi_s)\} : u \in \Omega, s \in [0, t]\}$$

then we have

$$A = \bigcup_{u \in \Omega} (A_u \cap \{\xi_t = u\})$$

for some collection of sets $A_u \in \mathcal{F}_t$. Checking that the property is retained on taking countable unions or complements then entails that the same property holds for any $A \in \tilde{\mathcal{F}}_t$. It is then straightforward to show that if Y is an $\tilde{\mathcal{F}}_t$-simple function (a finite combination of indicator functions) then it has the representation (2.1). But any non-negative Y is an increasing limit of simple functions, and any Y is a difference of two non-negative functions, and it is an easy exercise using these facts to complete the proof.

Definition 2.2:
We define the probability measure $\tilde{\mathbb{P}}$ on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}_\infty)$, by setting

$$\tilde{\mathbb{P}}[X] = \mathbb{P}\left[\sum_{u\in N(t)} X_u \prod_{v<u} \frac{1}{A_v+1}\right] \tag{2.2}$$

for each $\tilde{\mathcal{F}}_t$-measurable X with representation (2.1).

The measure $\tilde{\mathbb{P}}$ is an extension of $\mathbb{P}$ in that $\mathbb{P} = \tilde{\mathbb{P}}|_{\mathcal{F}_\infty}$, since $\sum_{u\in N(t)} \prod_{v<u} \frac{1}{A_v+1} = 1$. It is well-known (see for example [11]) that $\tilde{\mathbb{P}}$ can be decomposed as follows:

- the spine's motion is a standard Brownian motion;
- the lifetime of a spine particle is exponentially distributed with parameter r, independent of its motion;
- at the fission time of node u on the spine, the single spine particle is replaced by $A_u + 1$ children, with A_u being chosen independently of everything else and distributed according to the random variable A;
- the new spine particle is chosen uniformly from the $1 + A_u$ children;
- each of the remaining A_u children gives rise to an independent branching Brownian motion which is not part of the spine and is determined by a copy of the original measure $\mathbb{P}$ shifted to the time and place of its birth.

In summary, the spine behaves, under $\tilde{\mathbb{P}}$, just like any other particle.

2.4 Martingales and a change of measure

As we mentioned briefly in the introduction, one justification of the spine setup is that for any non-negative martingale for Brownian motion, we are able to construct a related non-negative additive martingale for BBM.

Indeed, suppose that we are given a non-negative, mean one, $\mathcal{G}_t$-adapted martingale $(\zeta(t), t \geq 0)$. (Since the path of the spine is simply a standard Brownian motion, we may use any normalised non-negative martingale for Brownian motion.) We call this martingale the *single-particle martingale*.

Definition 2.3:
We define an $\tilde{\mathcal{F}}_t$-adapted (and, in fact, $\tilde{\mathcal{G}}_t$-adapted) process $\tilde{\zeta}$ by

$$\tilde{\zeta}(t) = e^{-mrt}\zeta(t)\prod_{u<\xi_t}(A_u+1),$$

and an $\mathcal{F}_t$-adapted process Z by

$$Z(t) = \sum_{u\in N(t)} e^{-mrt}\zeta_u(t),$$

where ζ_u is the $\mathcal{F}_t$-adapted process defined via the representation of ζ as in (2.1). We call Z the *branching-particle martingale.*

We remark here that Z and ζ are, in fact, simply the projections of $\tilde{\zeta}$ onto the relevant filtrations:

- $Z(t) = \tilde{\mathbb{P}}[\tilde{\zeta}(t)|\mathcal{F}_t]$
- $\zeta(t) = \tilde{\mathbb{P}}[\tilde{\zeta}(t)|\mathcal{G}_t]$.

This will be implicit in the proof of Theorem 2.4 below.

Theorem 2.4:
Both $\tilde{\zeta}$ and Z are unit mean martingales on their respective filtrations.

Proof. Using the fact that the spine's motion is independent of its fission events,

$$\begin{aligned}
\tilde{\mathbb{P}}[\tilde{\zeta}(t)] &= \tilde{\mathbb{P}}\left[e^{-mrt}\zeta(t)\prod_{u<\xi_t}(A_u+1)\right] \\
&= e^{-mrt}\tilde{\mathbb{P}}\left[\tilde{\mathbb{P}}\left[\prod_{u<\xi_t}(A_u+1)\middle|\,\mathcal{G}_t\right]\zeta(t)\right] \\
&= e^{-mrt}\tilde{\mathbb{P}}\left[\prod_{u<\xi_t}(A_u+1)\right]\tilde{\mathbb{P}}[\zeta(t)] \\
&= e^{-mrt}\tilde{\mathbb{P}}\left[\prod_{u<\xi_t}(A_u+1)\right].
\end{aligned}$$

Now since the fission times are independent of the number of children at these times, we may condition first on knowing the fission times to see that

$$\tilde{\mathbb{P}}\left[\prod_{u<\xi_t}(A_u+1)\right]=\tilde{\mathbb{P}}\left[\prod_{u<\xi_t}\tilde{\mathbb{P}}[A_u+1]\right]=\tilde{\mathbb{P}}[(m+1)^{n(\xi_t)}]$$

where $n(\xi_t)$ is the generation of the spine particle at time t. Since this generation is a Poisson random variable with mean rt, we have $\tilde{\mathbb{P}}[(m+1)^{n(\xi_t)}]=e^{mrt}$ and hence

$$\tilde{\mathbb{P}}[\tilde{\zeta}(t)]=1.$$

We may now apply the Markov property to deduce that $\tilde{\zeta}(t)$ is a martingale with respect to $\tilde{\mathcal{F}}_t$. To see that Z is a martingale with respect to $\mathcal{F}_t$, we simply note that (using the representation (2.1) for $\zeta(t)$)

$$\begin{aligned}
\tilde{\mathbb{P}}[\tilde{\zeta}(t)|\mathcal{F}_t]&=\tilde{\mathbb{P}}\left[e^{-mrt}\zeta(t)\prod_{v<\xi_t}(A_v+1)\middle|\mathcal{F}_t\right]\\
&=\tilde{\mathbb{P}}\left[e^{-mrt}\sum_{u\in N(t)}\zeta_u(t)\mathbb{1}_{\xi_t=u}\prod_{v<\xi_t}(A_v+1)\middle|\mathcal{F}_t\right]\\
&=e^{-mrt}\sum_{u\in N(t)}\zeta_u(t)\prod_{v<u}(A_v+1)\tilde{\mathbb{P}}(\xi_t=u|\mathcal{F}_t)\\
&=e^{-mrt}\sum_{u\in N(t)}\zeta_u(t)\prod_{v<u}(A_v+1)\prod_{w<u}\frac{1}{A_w+1}\\
&=Z(t)
\end{aligned}$$

and the martingale property immediately follows. □

Definition 2.5:
We define a new probability measure, $\tilde{\mathbb{Q}}$, via

$$\left.\frac{d\tilde{\mathbb{Q}}}{d\tilde{\mathbb{P}}}\right|_{\tilde{\mathcal{F}}_t}:=\tilde{\zeta}(t).$$

Also, for convenience, define $\mathbb{Q}$ to be the projection of the measure $\tilde{\mathbb{Q}}$ onto $\mathcal{F}_\infty$; then

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}=Z(t).$$

Lemma 2.6:
Under $\tilde{\mathbb{Q}}$*,*

- *the spine moves as if under the changed measure given by*

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{G}_t} := \frac{\zeta(t)}{\zeta(0)}$$

where P is the law of a standard Brownian motion;

- *spine particles die at an accelerated rate $(1+m)r$ independently of their position;*

- *on death, a spine particle u is replaced (independently of its position and lifetime) by $1+A_u$ particles where the distribution of $1+A_u$ is size-biased: $\tilde{\mathbb{Q}}(A_u = k) = \frac{(1+k)\tilde{\mathbb{P}}(A=k)}{1+m}$;*

- *a new spine particle is chosen uniformly at random from the $1+A_u$ children at the fission point;*

- *the remaining child gives rise to an independent subtree, which is not part of the spine and is determined by an independent copy of the original measure $\mathbb{P}$ shifted to the position and time of creation.*

We saw a similar description of $\tilde{\mathbb{Q}}$ in Chapter 1 — this was originally given by Chauvin and Rouault in [5], where they made the key observation that the BBM is largely unchanged and that the only changes occur along the spine. The more advanced formulation above allows us to see explicitly which particle is the spine.

Hardy and Harris [11] develop the theory to cover more general branching processes. In Chapter 4 we shall also consider a case with a branching rate that depends on the position of the particle. For now however we prefer to convey the basic setup in order to avoid proofs cluttered with notation.

2.5 Spine tools

As we saw in Theorem 1.7, the pathwise construction of the changed measure $\tilde{\mathbb{Q}}$ seen above is not the only advantage of the spine theory. There are several other tools that will be extremely useful to us. The first, and perhaps most important, of these is the spine decomposition theorem. It is vital in that it allows us to relate the growth of the whole process to just the behaviour along the spine. This proof is taken from Hardy and Harris [11].

Theorem 2.7 (Spine decomposition)**:**
We have the following decomposition of the branching-particle martingale:

$$\tilde{\mathbb{Q}}[Z(t)|\tilde{\mathcal{G}}_\infty] = \sum_{u<\xi_t} A_u e^{-mrS_u}\zeta(S_u) + e^{-mrt}\zeta(t).$$

Proof. Since exactly one particle in $N(t)$ is the spine,

$$Z(t) = \sum_{\substack{u\in N(t)\\ u\neq\xi_t}} e^{-mrt}\zeta_u(t) + e^{-mrt}\zeta(t).$$

Each particle not in the spine has a unique ancestor in the spine, and so we partition the sum into the subtrees born at each fission point along the spine:

$$Z(t) = \sum_{u<\xi_t} e^{-mrS_u} \sum_{\substack{j=1,\dots,A_u+1\\ uj\notin\xi}} Z_{uj}(S_u;t) + e^{-mrt}\zeta(t)$$

where

$$Z_{uj}(S_u;t) := \sum_{\substack{v\in N(t)\\ u<v}} e^{-mr(t-S_u)}\zeta_v(t).$$

Now under $\tilde{\mathbb{Q}}$, conditional on $\tilde{\mathcal{G}}_{S_u}$, since the non-spine children of u draw out independent subtrees determined by copies of the original measure $\mathbb{P}$, we see that Z_{uj} is a $(\tilde{\mathbb{Q}}, \mathcal{F}_t, t \geq S_u)$-martingale on $[S_u, \infty)$ with initial value $Z_{uj}(S_u) = \zeta(S_u)$. Thus by the optional stopping theorem, for any $t \geq S_u$

$$\tilde{\mathbb{Q}}[Z_{uj}(S_u;t)|\mathcal{G}_\infty] = \tilde{\mathbb{Q}}[Z_{uj}(S_u;t)|\mathcal{G}_{S_u}] = \zeta(S_u)$$

and hence

$$\begin{aligned}\tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty] &= \sum_{u<\xi_t} e^{-mrS_u} \sum_{\substack{j=1,\dots,A_u+1\\ uj\notin\xi}} \zeta(S_u) + e^{-mrt}\zeta(t)\\ &= \sum_{u<\xi_t} A_u e^{-mrS_u}\zeta(S_u) + e^{-mrt}\zeta(t)\end{aligned}$$

as required. □

The spine decomposition is usually used in conjunction with the following result (usually we use $\mu = \mathbb{Q}$ and $\nu = \mathbb{P}$, so $X_t = Z(t)$). The proof is taken from [33].

Lemma 2.8:

Suppose that μ is a finite measure and ν a probability measure on the same measurable space $(\Omega, \mathcal{F})$, and that $\mathcal{F}_t$, $t \geq 0$ are increasing sub-σ-fields whose union generates $\mathcal{F}$. If, for each t, $\mu|_{\mathcal{F}_t}$ is absolutely continuous with respect to $\nu|_{\mathcal{F}_t}$ with Radon-Nikodým derivative X_t, and $X := \limsup X_t$, then for any $A \in \mathcal{F}$

$$\mu(A) = \nu[X \mathbb{1}_A] + \mu(A \cap \{X = \infty\}).$$

This entails

$$\mu \ll \nu \quad \Leftrightarrow \quad X < \infty \;\; \mu\text{-a.s.} \quad \Leftrightarrow \quad \mu = X\nu$$

and

$$\mu \perp \nu \quad \Leftrightarrow \quad X = \infty \;\; \mu\text{-a.s.} \quad \Leftrightarrow \quad \nu[X] = 0.$$

Proof. Define a new probability measure $\rho := (\mu + \nu)/C$ where $C := \mu(\Omega) + \nu(\Omega)$. Then $\mu \ll \rho$; set $U_t := (d\mu/d\rho)|_{\mathcal{F}_t}$ and $U := d\mu/d\rho$. Then $\mu[U|\mathcal{F}_t] = U_t$ so by the martingale convergence theorem $U = \lim_{t\to\infty} U_t$ ρ-almost surely. Similarly, setting $V_t := (d\nu/d\rho)|_{\mathcal{F}_t}$ and $V := d\nu/d\rho$ we have $V = \lim_{t\to\infty} V_t$ ρ-almost surely. Since for any $t \geq 0$, $U_t + V_t = C$ ρ-almost surely, we must have $\rho(U = V = 0) = 0$, and so (ρ-almost surely)

$$\frac{U}{V} = \frac{\lim_{t\to\infty} U_t}{\lim_{t\to\infty} V_t} = \lim_{t\to\infty} \frac{U_t}{V_t} = \lim_{t\to\infty} X_t = X.$$

Thus for any $A \in \mathcal{F}$

$$\begin{aligned} \mu(A) = \rho[U\mathbb{1}_A] &= \rho[XV\mathbb{1}_A] + \rho[U\mathbb{1}_{V=0}\mathbb{1}_A] \\ &= \nu[X\mathbb{1}_A] + \mu(A \cap \{V = 0\}) \\ &= \nu[X\mathbb{1}_A] + \mu(A \cap \{X = \infty\}). \end{aligned}$$

□

Our final theorem in this section is a many-to-one theorem. Similar theorems have been known for much longer than the spine theory, but the spine allows us a simple and intuitive proof. We first prove an interesting lemma, taken from Hardy and Harris [11].

Lemma 2.9:
For any label $u \in \Omega$ and $t \geq 0$,

$$\tilde{\mathbb{Q}}(\xi_t = u|\mathcal{F}_t) = \frac{e^{-mrt}\zeta_u(t)}{Z(t)} \mathbb{1}_{\{u \in N(t)\}}.$$

Proof. For any $F \in \mathcal{F}_t$,

$$\begin{aligned}
\tilde{\mathbb{Q}}(\{\xi_t = u\} \cap F) &= \tilde{\mathbb{P}}[\mathbb{1}_{\{\xi_t=u\}}\mathbb{1}_F \prod_{v<\xi_t}(1+A_v)e^{-mrt}\zeta(t)] \\
&= \tilde{\mathbb{P}}[\mathbb{1}_{\{\xi_t=u\}}\mathbb{1}_F \sum_{w\in N(t)}\prod_{v<w}(1+A_v)e^{-mrt}\zeta_w(t)\mathbb{1}_{\{\xi_t=w\}}] \\
&= \tilde{\mathbb{P}}[\mathbb{1}_{\{u\in N(t)\}}\mathbb{1}_F \prod_{v<u}(1+A_v)e^{-mrt}\zeta_u(t)\mathbb{1}_{\{\xi_t=u\}}] \\
&= \mathbb{P}[\mathbb{1}_{u\in N(t)}e^{-mrt}\zeta_u(t)\mathbb{1}_F] \\
&= \mathbb{Q}\left[\frac{1}{Z(t)}\mathbb{1}_{\{u\in N(t)\}}e^{-mrt}\zeta_u(t)\mathbb{1}_F\right]
\end{aligned}$$

where for the last equality we have used the fact that if $u \in N(t)$ and $\zeta_u(t) > 0$ then $Z(t) > 0$. □

We come now to the many-to-one theorem. We use a relatively simple form compared to that in [11]. As an example of its use, one might imagine applying it to the function

$$g(t) := \mathbb{1}_{\{\xi_t\in A\}} = \sum_{u\in N(t)} \mathbb{1}_{\{X_u(t)\in A\}}\mathbb{1}_{\{\xi_t=u\}}$$

for some set $A \subseteq \mathbb{R}$ in order to calculate the expected number of particles within A at time t.

Theorem 2.10 (Many-to-one theorem)**:**
If $g(t)$ is $\mathcal{G}_t$-measurable and is written

$$g(t) = \sum_{u\in N(t)} g_u(t)\mathbb{1}_{\{\xi_t=u\}}$$

where each $g_u(t)$ is $\mathcal{F}_t$-measurable, then

$$\mathbb{P}\left[\sum_{u\in N(t)} g_u(t)\right] = \tilde{\mathbb{P}}[e^{mrt}g(t)].$$

Proof. We use the spine theory with the single-particle martingale $\zeta(t) \equiv 1$. By the fact that $\tilde{\mathbb{Q}}|_{\mathcal{G}_t} = \zeta(t)\tilde{\mathbb{P}}|_{\mathcal{G}_t}$,

$$\begin{aligned}\tilde{\mathbb{P}}[e^{mrt}g(t)] &= \tilde{\mathbb{P}}[\zeta(t)e^{mrt}g(t)] \\ &= \tilde{\mathbb{Q}}[e^{mrt}g(t)] \\ &= \tilde{\mathbb{Q}}[e^{mrt}\sum_{u\in N(t)} g_u(t)\mathbb{1}_{\{\xi_t=u\}}] \\ &= \tilde{\mathbb{Q}}[e^{mrt}\sum_{u\in N(t)} g_u(t)\tilde{\mathbb{Q}}(\xi_t = u|\mathcal{F}_t)].\end{aligned}$$

Applying Lemma 2.9,

$$\tilde{\mathbb{P}}[e^{mrt}g(t)] = \tilde{\mathbb{Q}}\left[e^{mrt}\sum_{u\in N(t)} g_u(t)\frac{e^{-mrt}}{Z(t)}\right] = \tilde{\mathbb{Q}}\left[\frac{1}{Z(t)}\sum_{u\in N(t)} g_u(t)\right].$$

But $\sum_{u\in N(t)} g_u(t)/Z(t)$ is $\mathcal{F}_t$-measurable, so by the fact that $\tilde{\mathbb{Q}}|_{\mathcal{F}_t} = Z(t)\tilde{\mathbb{P}}|_{\mathcal{F}_t} = Z(t)\mathbb{P}|_{\mathcal{F}_t}$,

$$\tilde{\mathbb{P}}[e^{mrt}g(t)] = \mathbb{P}\left[\sum_{u\in N(t)} g_u(t)\right]. \qquad \square$$

Chapter 3

Measure changes with extinction

For this chapter, rather than considering only spine changes of measure, we generalise and consider any unit-mean martingale change of measure $\mathbb{Q}|_{\mathcal{F}_t} := Z_t\mathbb{P}|_{\mathcal{F}_t}$. We clarify that in general $1/Z_t$ is only a supermartingale under $\mathbb{Q}$, and then give a necessary and sufficient condition for the identity $\mathbb{P}(\exists t : Z_t = 0) = \mathbb{P}(Z_\infty = 0)$ to hold. This work is joint with Simon Harris and appears in [18].

3.1 Introduction

Consider two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on the same filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ along with a càdlàg adapted non-negative process (Z_t) such that, for each t,

$$\mathbb{Q}\big|_{\mathcal{F}_t} = Z_t\mathbb{P}\big|_{\mathcal{F}_t}.$$

The process Z may be in either continuous (usually $t \in \mathbb{R}_+$) or discrete (usually $t \in \mathbb{Z}_+$) time; we shall not always distinguish between the two. It is easy to see that Z is necessarily a non-negative $\mathbb{P}$-martingale with unit mean. We define

$$\Upsilon := \inf\{t \geq 0 : Z_t = 0\};$$

we call this the extinction time of the process Z.

It has been claimed, in particular in Biggins and Kyprianou [3], Englander and Kyprianou [7] and Lyons [31], that the process $1/Z_t$ is automatically a $\mathbb{Q}$-martingale. This is not always true, as shown in the example below. However, in Proposition 3.2 we show that $1/Z_t$ is a supermartingale. Since the proofs in [3], [7] and [31] depend only on showing that $1/Z_t$ converges $\mathbb{Q}$-almost surely, the supermartingale property is sufficient and their results are unaffected.

Example 3.1:
Consider the (discrete time) Galton-Watson process in which each particle has either 2 children, with probability p, or no children, with probability $q = 1 - p$. Let X_n be the number of particles in the nth generation, and set

$$m = 2p \quad \text{and} \quad Z_n = X_n/m^n.$$

It is well-known that Z is a $\mathbb{P}$-martingale. Making the change of measure to $\mathbb{Q}$, we can check immediately that

$$\mathbb{Q}(Z_1 = 0) = \mathbb{P}[Z_1 \mathbb{1}_{\{Z_1=0\}}] = 0,$$

so

$$\begin{aligned} \mathbb{Q}[1/Z_1] &= m\sum_{j=1}^{\infty} \mathbb{Q}(X_1 = j)/j = m\sum_{j=1}^{\infty} \mathbb{P}[Z_1 \mathbb{1}_{\{X_1=j\}}]/j \\ &= m(2/2m)\mathbb{P}(X_1 = 2) = \mathbb{P}(X_1 = 2) = p. \end{aligned}$$

Since $\mathbb{Q}[1/Z_0] = 1$, we see that $(1/Z_n)$ is not a $\mathbb{Q}$-martingale if $p < 1$.

In fact we show in Lemma 3.4 that in all cases, for any $t \in [0, \infty)$,

$$\mathbb{Q}[1/Z_t] = \mathbb{P}(Z_t > 0) = \mathbb{P}(\Upsilon > t)$$

and in Theorem 3.5 we see that the identity

$$\mathbb{Q}[1/Z_\infty] = \mathbb{P}(Z_\infty > 0) = \mathbb{P}(\Upsilon = \infty)$$

holds if and only if $1/Z_t$ is uniformly integrable. Such results, linking the extinction of the process to the event that the martingale limit is zero, are often of great value in the branching process scenario. We stress, however, that all of our results apply to general measure changes rather than just those related to branching processes.

3.2 Main results

3.2.1 The $\mathbb{Q}$-supermartingale property of $1/Z$

We may easily show that, as claimed earlier, $1/Z_t$ is a $\mathbb{Q}$-supermartingale.

Proposition 3.2:

$$\mathbb{Q}\left[\frac{1}{Z_{t+s}}\middle| \mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{P}(Z_{t+s} > 0 \mid \mathcal{F}_t).$$

In particular, $1/Z_t$ is a $\mathbb{Q}$-supermartingale.

Proof. First, note that there is no extinction under $\mathbb{Q}$: for all $t > 0$,

$$\mathbb{Q}(Z_t = 0) = \mathbb{P}[Z_t \mathbb{1}_{\{Z_t=0\}}] = 0.$$

Also, there is no rebirth after extinction; that is, for all $s, t > 0$,

$$Z_t = 0 \Rightarrow Z_{t+s} = 0 \quad \text{(a.s. under } \mathbb{P}\text{)}.$$

This fact can be shown directly, using the martingale property of Z; however, the measure change allows us a simple proof:

$$\mathbb{P}(Z_{t+s} > 0, Z_t = 0) = \mathbb{P}\left[\frac{Z_{t+s}}{Z_{t+s}} \mathbb{1}_{\{Z_{t+s}>0, Z_t=0\}}\right] = \mathbb{Q}\left[\frac{1}{Z_{t+s}} \mathbb{1}_{\{Z_{t+s}>0, Z_t=0\}}\right] = 0,$$

since $\mathbb{Q}(Z_t = 0) = 0$. Using these two facts, we see that for any $A \in \mathcal{F}_t$,

$$\begin{aligned}
\mathbb{Q}\left[\frac{1}{Z_t}\mathbb{P}(Z_{t+s} > 0|\mathcal{F}_t)\mathbb{1}_A\right] &= \mathbb{Q}\left[\frac{1}{Z_t}\mathbb{1}_{\{Z_t>0\}}\mathbb{P}(Z_{t+s} > 0|\mathcal{F}_t)\mathbb{1}_A\right] \\
&= \mathbb{P}\left[\frac{Z_t}{Z_t}\mathbb{1}_{\{Z_t>0\}}\mathbb{P}(Z_{t+s} > 0|\mathcal{F}_t)\mathbb{1}_A\right] \\
&= \mathbb{P}(Z_t > 0,\ Z_{t+s} > 0,\ A) \;=\; \mathbb{P}(Z_{t+s} > 0,\ A) \\
&= \mathbb{P}\left[\frac{Z_{t+s}}{Z_{t+s}}\mathbb{1}_{\{Z_{t+s}>0\}}\mathbb{1}_A\right] \;=\; \mathbb{Q}\left[\frac{1}{Z_{t+s}}\mathbb{1}_A\right].
\end{aligned}$$

Thus, by definition of conditional expectation,

$$\mathbb{Q}\left[\frac{1}{Z_{t+s}}\middle|\mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{P}(Z_{t+s} > 0|\mathcal{F}_t). \qquad \square$$

Remark:
Kuhlbusch [26] gives a very similar proof of this fact, albeit in discrete time only. The proof above also has the advantage that it gives an explicit formula for the rate at which the process is decaying.

Corollary 3.3:
$(1/Z_t)$ is a true $\mathbb{Q}$-martingale if and only if there is no extinction under $\mathbb{P}$.

3.2.2 Extinction probabilities

In work on branching processes, extinction probabilities often cause difficulties. For example, recall that we defined Υ to be the extinction time,

$$\Upsilon := \inf\{t : Z_t = 0\}$$

and set

$$Z_\infty := \limsup_{t\to\infty} Z_t;$$

then it can be a major problem to prove that

$$\mathbb{P}(Z_\infty > 0) = \mathbb{P}(\Upsilon = \infty). \tag{3.1}$$

We give an identity suited to this purpose. Despite its simplicity, it can be extremely useful – for example it is an essential ingredient in the proofs of Chapter 5.

Lemma 3.4:
For any $t \in [0, \infty)$*,*

$$\mathbb{P}(\Upsilon > t) = \mathbb{P}(Z_t > 0) = \mathbb{Q}[1/Z_t];$$

also

$$\mathbb{P}(Z_\infty > 0) = \mathbb{Q}[1/Z_\infty].$$

Proof. Using various facts from earlier,

$$\mathbb{Q}[1/Z_t] = \mathbb{Q}\left[\frac{1}{Z_t}\mathbb{1}_{\{Z_t>0\}}\right] = \mathbb{P}\left[Z_t\frac{1}{Z_t}\mathbb{1}_{\{Z_t>0\}}\right] = \mathbb{P}(Z_t > 0)$$

which establishes the first equality. For the second, we use Lemma 2.8. Note that

$$\mathbb{Q}(Z_\infty = 0) = \mathbb{P}[Z_\infty \mathbb{1}_{\{Z_\infty=0\}}] + \mathbb{Q}(\{Z_\infty = 0\} \cap \{Z_\infty = \infty\}) = 0.$$

Thus, using Lemma 2.8 again,

$$\mathbb{Q}[1/Z_\infty] = \mathbb{Q}\left[\frac{1}{Z_\infty}\mathbb{1}_{\{Z_\infty>0\}}\right] = \mathbb{P}(Z_\infty > 0) + \mathbb{Q}\left[\frac{1}{Z_\infty}\mathbb{1}_{\{Z_\infty=\infty\}}\right] = \mathbb{P}(Z_\infty > 0). \qquad \square$$

This allows us to give a simple necessary and sufficient condition for (3.1) to hold.

Theorem 3.5:
The full identity

$$\mathbb{Q}[1/Z_\infty] = \mathbb{P}(Z_\infty > 0) = \mathbb{P}(\Upsilon = \infty)$$

holds if and only if the set $\{1/Z_t : t \geq 0\}$ *is* $\mathbb{Q}$*-uniformly integrable.*

Proof. If $\{1/Z_t : t > 0\}$ is $\mathbb{Q}$-uniformly integrable then we have immediately that

$$\mathbb{P}(Z_\infty > 0) = \mathbb{Q}[1/Z_\infty] = \lim_{t\to\infty} \mathbb{Q}[1/Z_t] = \lim_{t\to\infty} \mathbb{P}(\Upsilon > t) = \mathbb{P}(\Upsilon = \infty).$$

Conversely, if $\mathbb{P}(Z_\infty > 0) = \mathbb{P}(\Upsilon = \infty)$, then as above we have

$$\mathbb{Q}[1/Z_\infty] = \lim_{t\to\infty} \mathbb{Q}[1/Z_t].$$

Thus (by Scheffé's lemma – Theorem 5.10 of [37]) $1/Z_t$ converges in L^1 to $1/Z_\infty$. Convergence in L^1 then implies uniform integrability (see Theorem 13.7 of [37] for example); hence $\{1/Z_t : t \geq 0\}$ is $\mathbb{Q}$-uniformly integrable. □

3.3 The $\mathbb{Q}$-local martingale property

We may now ask whether $(1/Z_t, t \geq 0)$ is even a $\mathbb{Q}$-local martingale. The intuition is that if, as is often the case, Z_t is some suitable rescaling of the number of particles alive at time t, then $1/Z_t$ is perfectly well-behaved under $\mathbb{Q}$: there is always at least one particle alive, so Z_t cannot get within a certain distance of zero. Thus $1/Z_t$ can only be a local martingale if it is a true martingale; but it is not a true martingale, and thus not a local martingale.

This notion is made precise in Proposition 3.7 below. The result is really just a rephrasing of a standard fact about local martingales, which we state in Lemma 3.6; we give a proof of Proposition 3.7 regardless.

Lemma 3.6:
Suppose that $(X_t, t \geq 0)$ is a local martingale. Then the following are equivalent:

- *X is a martingale;*
- *For each $t > 0$, $\{X_T : T$ is a stopping time, $T \leq t\}$ is uniformly integrable.*

Proposition 3.7:
Suppose that extinction occurs with positive probability under $\mathbb{P}$, i.e. there exists $s > 0$ such that $\mathbb{P}(Z_s = 0) > 0$, and that the set

$$\{1/Z_T : T \text{ is a stopping time, } T \leq t\}$$

is $\mathbb{Q}$-UI for each $t > 0$. Then $1/Z_t$ is not a local martingale under $\mathbb{Q}$.

Proof. For a contradiction, suppose that $1/Z_t$ is a local martingale under $\mathbb{Q}$, with a reducing sequence of stopping times $(T_n, n \geq 0)$. Then for any bounded stopping time

$T \leq t$, say,

$$\mathbb{Q}[1/Z_0] = \mathbb{Q}[1/Z_0^{T_n}] = \mathbb{Q}[1/Z_T^{T_n}] = \mathbb{Q}[1/Z_{T\wedge T_n}],$$

where the second equality holds by the optional stopping theorem. Now by hypothesis $\{Z_{T\wedge T_n} : n \geq 0\}$ is UI and thus

$$\mathbb{Q}[1/Z_{T\wedge T_n}] \to \mathbb{Q}[1/Z_T] \quad \text{as} \quad n \to \infty.$$

So $\mathbb{Q}[1/Z_T] = \mathbb{Q}[1/Z_0]$ for all bounded stopping times T, and hence by optional stopping $1/Z_t$ is a true $\mathbb{Q}$-martingale. We have already shown that this is not true when there is a positive probability of extinction (Corollary 3.3); hence by contradiction $1/Z_t$ is not a $\mathbb{Q}$-local martingale. □

Example 3.8:
Consider a standard branching Brownian motion with branching rate r and birth distribution A taking values in $\{-1, 1, 2, 3, \ldots\}$ with $\mathbb{P}[A] = m \in (0, \infty)$. Let $N(t)$ be the set of particles at time t, with particle u having position $X_u(t)$. Then, as in Section 1.2, it is known that

$$Z_\lambda(t) := \sum_{u\in N_t} e^{-mrt+\lambda X_u(t)-\lambda^2 t/2}$$

is a martingale. Suppose that $\mathbb{P}(A = -1) > 0$. Then making the usual change of measure to $\mathbb{Q}$, we know that $(1/Z_\lambda(t), t \geq 0)$ is not a $\mathbb{Q}$-martingale. It is possible, by using the spine interpretation of the measure change, to show that it is not even a local martingale. We embellish our probability space as in Chapter 2 with extra information concerning one distinguished infinite line of descent, called the spine, and define a new measure $\tilde{\mathbb{Q}}$ which is an extension of $\mathbb{Q}$. Under $\tilde{\mathbb{Q}}$ the spine moves with a drift λ, and the birth rate along the spine is also altered. The spine almost surely survives forever under $\tilde{\mathbb{Q}}$, and we denote its position at time t by ξ_t. Thus almost surely under $\tilde{\mathbb{Q}}$, for a bounded stopping time $T \leq t$ say,

$$\begin{aligned}
\frac{1}{Z_\lambda(T)} &= \frac{1}{\sum_{u\in N(T)} e^{-rT+\lambda X_u(T)-\lambda^2 T/2}} \\
&\leq \frac{1}{e^{-rT+\lambda\xi_T-\lambda^2 T/2}} \\
&= e^{(r+\lambda^2)T} \cdot e^{-\lambda(\xi_T-\lambda T)-\lambda^2 T/2} \\
&\leq e^{(r+\lambda^2)t} \cdot e^{-\lambda(\xi_T-\lambda T)-\lambda^2 T/2}.
\end{aligned}$$

Since $(e^{-\lambda(\xi_t-\lambda t)-\lambda^2 t/2}, t \geq 0)$ is a martingale under $\tilde{\mathbb{Q}}$ (because ξ is a Brownian motion with drift λ), by Lemma 3.6 the set

$$\{e^{-\lambda(\xi_T-\lambda T)-\lambda^2 T/2} : \, T \text{ is a stopping time}, \, T \leq t\}$$

is $\tilde{\mathbb{Q}}$-uniformly integrable. Multiplying each element of the set by a constant $e^{(r+\lambda^2)t}$ does not change this property, and hence by domination

$$\{1/Z_\lambda(T) : \, T \text{ is a stopping time}, \, T \leq t\}$$

is uniformly integrable under $\tilde{\mathbb{Q}}$ (and so under $\mathbb{Q}$). Proposition 3.7 now tells us that $1/Z_\lambda(t)$ is not a local martingale under $\mathbb{Q}$.

Chapter 4

Branching Brownian motion: Scaled growth along paths in an inhomogeneous branching environment

We consider a branching Brownian motion in which each particle breeds at a rate depending on its position, giving birth to a random number of offspring. We give a result on the growth of the number of particles along chosen paths in this scenario. The work follows the approach of classical large deviations results, in which paths in $C[0,1]$ are rescaled onto $C[0,T]$ for large T. The methods used are probabilistic and take advantage of spine techniques as seen in Chapter 2. This work is a generalisation of the article [17].

4.1 Introduction and statement of result

4.1.1 Introduction

Fix $\beta > 0$, $p \in [0,2)$ and a random variable A taking values in $\mathbb{N}$ such that $m := E[A] \in (1,\infty)$ and $E[A \log_+ A] < \infty$. Consider a branching Brownian motion (BBM) under a probability measure $\mathbb{P}$, starting with one particle at the origin and in which each particle u, once born, performs a Brownian motion independent of all other particles until it dies, an event which occurs with probability $\beta|x|^p dt + o(dt)$ if the particle is in position x at time t. At its time of death each particle is replaced by a random number $1 + A_u$ of offspring where A_u has the same distrubition as A. We let $N(t)$ be the set of particles alive at time t. For $u \in N(t)$ let $X_u(t)$ be the position of particle u at time t and extend this concept to times $s \le t$ by setting $X_u(s) := X_v(s)$ if $v \in N(s)$ and v is an ancestor of u.

Fix a set $D \subseteq C[0,1]$ and $\theta \in [0,1]$, and let $q := \frac{2}{2-p}$; then we are interested in the size of the sets

$$N_T(D,\theta) := \{u \in N(\theta T) : \exists f \in D \text{ with } X_u(t) = T^q f(t/T) \ \ \forall t \in [0, \theta T]\}$$

for large T.

4.1.2 The main result

We define the class H_1 of functions by

$$H_1 := \left\{ f \in C[0,1] : \exists g \in L^2[0,1] \text{ with } f(s) = \int_0^s g(s)ds \ \ \forall s \in [0,1] \right\},$$

and to save on notation we set $f'(t) := \infty$ if $f \in C[0,1]$ is not differentiable at the point t. We then take integrals in the Lebesgue sense so that we may integrate functions that equal ∞ on sets of zero measure. We let

$$\theta_0(f) := \inf \left\{ \theta \in [0,1] : m\beta \int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds < 0 \right\} \ \in [0,1] \cup \{\infty\}$$

(we think of θ_0 as the extinction time along f, the time at which the number of particles near f hits zero) and define our rate function K, for $f \in C[0,1]$ and $\theta \in [0,1]$, as

$$K(f,\theta) := \begin{cases} m\beta \int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds & \text{if } f \in H_1 \text{ and } \theta \le \theta_0(f) \\ -\infty & \text{otherwise.} \end{cases}$$

We expect approximately $\exp(K(f,\theta)T^{2q-1})$ particles whose paths up to time θT (when suitably rescaled) look like f. This is made precise in Theorem 4.1.

Theorem 4.1:
For any closed set $D \subseteq C[0,1]$ and $\theta \in [0,1]$,

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| \le \sup_{f \in D} K(f,\theta)$$

almost surely, and for any open set $A \subseteq C[0,1]$ and $\theta \in [0,1]$,

$$\liminf_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(A,\theta)| \ge \sup_{f \in A} K(f,\theta)$$

almost surely.

Sections 4.3 and 4.4 will be concerned with giving a proof of this theorem.

This theorem extends the result of Git [8] to the inhomogeneous branching potential introduced by Harris and Harris [14]. The methods used are similar to those in Harris and Roberts [17]: although there are new difficulties introduced by the position-dependent branching that require various (analytic and probabilistic) improvements to the arguments in [17], the probabilistic ideas at the heart of the proof remain the same.

Our tactic for the proof is to first work along lattice times, and then upgrade to the full result using Borel-Cantelli arguments. We begin, in Section 4.2, by introducing a family of martingales and changes of measure which will provide us with useful tools for our proofs. We then apply these tools in Section 4.3 to give a proof of the lower bound for Theorem 4.1, following a fairly straightforward heuristic argument. Finally, in Section 4.4, we prove the upper bound in Theorem 4.1 — it turns out that some slightly involved work-arounds are required to lift up the proof from [17] to our more general setting.

4.1.3 The oversight in Git [8]

In [8] it is written that under a certain assumption, setting

$$W_n = \left\{\omega \in \Omega : \limsup_{T\to\infty} \frac{1}{T} \log |N_T(D,\theta)| > J(D,\theta) + \frac{1}{n}\right\}$$

(it is not important what $J(D,\theta)$ is here) we have $\mathbb{P}(W_n) > 0$ for some n. This is correct, but the article then goes on to say "It is now clear that

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\big[|N_T(D,\theta)|\big] \geq J(D,\theta) + \frac{1}{n} \text{ "}$$

which does not appear to be obviously true. To see this explicitly, work on the probability space $[0,1]$ with Lebesgue probability measure $\mathbb{P}$. Let X_T, $T \geq 0$ be the càdlàg random process defined (for $\omega \in [0,1]$ and $T \geq 0$) by

$$X_T(\omega) = \begin{cases} e^{2T} & \text{if } T - n \in [\omega - e^{-4T},\ \omega + e^{-4T}) \text{ for some } n \in \mathbb{N} \\ e^{T} & \text{otherwise.} \end{cases}$$

Then for every ω,

$$\limsup \frac{1}{T} \log X_T(\omega) = 2$$

but

$$\frac{1}{T} \log \mathbb{E}[X_T] \to 1.$$

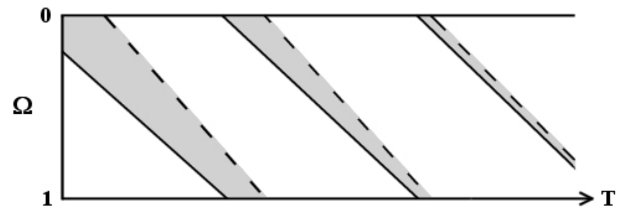

Figure 4-1: Visualisation of the process $X_T, T \geq 0$. Grey areas show where $X_T(\omega) = e^{2T}$, and white areas where $X_T(\omega) = e^T$.

4.2 A family of spine martingales

4.2.1 The spine setup

We will need to use some of the spine tools of Chapter 2 as part of our proof. However, as we are dealing with an inhomogeneous branching rate that was not covered in Chapter 2, we once more give a short description of the spine construction and state the results that we will need for this section. The proofs are essentially the same as those already given, and we refer the interested reader to Hardy and Harris' more general formulation in [12].

We first embellish our probability space by keeping track of some extra information about one particular infinite line of descent or *spine*. This line of descent is defined as follows: our original particle is part of the spine; when this particle dies, we choose one of its children uniformly at random to become part of the spine. We continue in this manner: when a spine particle dies, we choose one of its children uniformly at random to become part of the spine. In this way at any time $t \geq 0$ we have exactly one particle in $N(t)$ that is part of the spine. We refer to both this particle and its position with the label ξ_t; this is a slight abuse of notation, but it should always be clear from the context which meaning is intended. The spatial motion of the spine, $(\xi_t)_{t\geq 0}$, is a standard Brownian motion.

The resulting probability measure (on the set of *marked Galton-Watson trees with spines*) we denote by $\tilde{\mathbb{P}}$, and we find need for four different filtrations to encode differing amounts of this new information:

- $\mathcal{F}_t$ contains the all the information about the marked tree up to time t. However, it does not know which particle is the spine at any point. Thus it is simply the natural filtration of the original branching Brownian motion.

- $\tilde{\mathcal{F}}_t$ contains all the information about both the marked tree and the spine up to time t.

- $\tilde{\mathcal{G}}_t$ contains all the information about the spine up to time t, including the birth times of other particles along its path and how many children are born at each of these times; it does not know anything about the rest of the tree.
- $\mathcal{G}_t$ contains just the spatial information about the spine up to time t; it does not know anything about the rest of the tree.

We note that $\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t$ and $\mathcal{G}_t \subseteq \tilde{\mathcal{G}}_t \subseteq \tilde{\mathcal{F}}_t$, and also that $\tilde{\mathbb{P}}$ is an extension of $\mathbb{P}$ in that $\mathbb{P} = \tilde{\mathbb{P}}|_{\mathcal{F}_\infty}$.

Lemma 4.2 (Many-to-one lemma)**:**
If $g(t)$ is $\mathcal{G}_t$-measureable and can be written

$$g(t) = \sum_{u \in N(t)} g_u(t) \mathbb{1}_{\{\xi_t = u\}}$$

where each $g_u(t)$ is $\mathcal{F}_t$-measureable, then

$$\mathbb{E}\left[\sum_{u \in N(t)} g_u(t)\right] = \tilde{\mathbb{E}}[e^{m\beta \int_0^t |\xi_s|^p ds} g(t)].$$

This lemma is extremely useful as it allows us to reduce questions about the entire population down to calculations involving just one standard Brownian motion — the spine. A proof of a more general version of this lemma may be found in [12].

4.2.2 Martingales and changes of measure

For $f \in C[0,1]$, $\theta \in [0,1]$ and $\varepsilon > 0$, define

$$N_T(f, \varepsilon, \theta) := \{u \in N(\theta T) : |X_u(t) - T^q f(t/T)| < \varepsilon T^q \;\; \forall t \in [0, \theta T]\}$$

so that $N_T(f, \varepsilon, \theta) = N_T(B(f, \varepsilon), \theta)$. We look for martingales associated with these sets. For convenience, in this section we use the shorthand

$$N_T(t) := N_T(f, \varepsilon, t/T).$$

Lemma 4.3:
If $f \in C^2[0,1]$ then the process

$$V_T(t) := e^{\pi^2 t / 8\varepsilon^2 T^{2q}} \cos\left(\frac{\pi}{2\varepsilon T^q}(\xi_t - T^q f(t/T))\right) e^{T^{q-1} \int_0^t f'(s/T) d\xi_s - \frac{1}{2} T^{2q-1} \int_0^{t/T} f'(s)^2 ds},$$

$t \in [0, T]$, is a $\mathcal{G}_t$-local martingale under $\tilde{\mathbb{P}}$.

Proof. Since the motion of the spine is simply a standard Brownian motion under $\tilde{\mathbb{P}}$, we may apply Itô's formula (the sufficient conditions of, for example, Lawler [29] tell us that if $f \in C^2[0,1]$ then V_T is sufficiently smooth for Itô's formula to hold). Let

$$\tilde{V}_T(t) := e^{\pi^2 t/8\varepsilon^2 T^{2q}} \sin\left(\frac{\pi}{2\varepsilon T^q}(\xi_t - T^q f(t/T))\right) e^{T^{q-1}\int_0^t f'(s/T)d\xi_s - \frac{1}{2}T^{2q-1}\int_0^{t/T} f'(s)^2 ds}.$$

Then

$$\begin{aligned} dV_T(t) = &\frac{\pi^2}{8\varepsilon^2 T^{2q}} V_T(t)dt + \frac{\pi}{2\varepsilon T} f'(t/T)\tilde{V}_T(t)dt - \frac{1}{2}T^{2q-1} f'(t/T)^2 V_T(t)dt \\ &- \frac{\pi}{2\varepsilon T^q}\tilde{V}_T(t)d\xi_t + T^{q-1} f'(t/T)V_T(t)d\xi_t \\ &- \frac{\pi^2}{8\varepsilon^2 T^{2q}} V_T(t)dt + \frac{1}{2}T^{2q-1} f'(t/T)^2 V_T(t)dt - \frac{\pi}{2\varepsilon T} f'(t/T)\tilde{V}_T(t)dt \end{aligned}$$

which completes the proof. □

By stopping the process $(V_T(t), t \in [0,T])$ at the first exit time of the Brownian motion from the tube $\{(x,t) : |T^q f(t/T) - x| < \varepsilon T^q\}$, we obtain also that

$$\zeta_T(t) := V_T(t)\mathbb{1}_{\{|T^q f(s/T) - \xi_s| < \varepsilon T^q \ \forall s \le t\}}, \quad t \in [0,T]$$

is a non-negative $\mathcal{G}_t$-local martingale, and since its size is then clearly constrained it must (by Lemma 3.6) in fact be a $\mathcal{G}_t$-martingale. As in [12] (and analogously to the developments in Chapter 2), we may build from ζ_T a collection of $\tilde{\mathcal{F}}_t$-martingales $\tilde{\zeta}_T$ given by

$$\tilde{\zeta}_T(t) := \prod_{v < \xi_t}(1 + A_v)e^{-m\beta\int_0^t |\xi_s|^p ds}\zeta_T(t), \quad t \in [0,T].$$

When we project $\tilde{\zeta}_T(t)$ back onto $\mathcal{F}_t$ we get a new set of mean-one $\mathcal{F}_t$-martingales $(Z_T(t), t \ge 0)$.

These processes Z_T are the main objects of interest in this section, and can be expressed for $t \in [0,T]$ as the sum

$$Z_T(t) = \sum_{u \in N_T(t)} V_T^{(u)}(t)e^{-m\beta\int_0^t |X_u(s)|^p ds}$$

where

$$V_T^{(u)}(t) := e^{\pi^2 t/8\varepsilon^2 T^{2q}} \cos\left(\frac{\pi}{2\varepsilon T^q}(X_u(t) - T^q f(t/T))\right) \cdot e^{T^{q-1}\int_0^t f'(s/T)dX_u(s) - \frac{1}{2}T^{2q-1}\int_0^{t/T} f'(s)^2 ds}.$$

We now define new measures, $\tilde{\mathbb{Q}}_T$, via

$$\tilde{\mathbb{Q}}_T|_{\tilde{\mathcal{F}}_t} := \tilde{\zeta}_T(t)\tilde{\mathbb{P}}|_{\tilde{\mathcal{F}}_t}$$

for $t \in [0,T]$ — and note that

$$\tilde{\mathbb{Q}}_T|_{\mathcal{F}_t} = Z_T(t)\tilde{\mathbb{P}}|_{\mathcal{F}_t} \quad \text{and} \quad \tilde{\mathbb{Q}}_T|_{\mathcal{G}_t} = \zeta_T(t)\tilde{\mathbb{P}}|_{\mathcal{G}_t}.$$

Lemma 4.4:
Under $\tilde{\mathbb{Q}}_T$, the spine $(\xi_t, t \in [0,T])$ moves as a Brownian motion with drift

$$T^{q-1}f'(t/T) - \frac{\pi}{2\varepsilon T^q}\tan\left(\frac{\pi}{2\varepsilon T^q}(x - T^q f(t/T))\right)$$

when at position x at time t; in particular,

$$|\xi_t - T^q f(t/T)| \leq \varepsilon T^q \ \ \forall t \leq T.$$

Each particle u in the spine dies at an accelerated rate $m\beta|x|^p$ when in position x, to be replaced by a random number $A_u + 1$ of offspring where A_u is taken from the size-biased *distribution relative to A, given by $\tilde{\mathbb{Q}}_T(A_u = k) = (m+1)^{-1}(k+1)P(A = k)$ (note that this distribution does not depend on T). All non-spine particles, once born, behave* exactly as they would under $\mathbb{P}$*: they move like independent standard Brownian motions, die at the normal rate $\beta|x|^p$, and give birth to a number of particles that is distributed like $1 + A$.*

Proof. A proof of this result can be found in [12], and again we saw an analogous result in the homogeneous breeding case in Chapter 2. We will not use the precise drift of the spine except for the fact that it remains within the tube: to see this note that since the event is $\tilde{\mathcal{G}}_T$-measurable,

$$\tilde{\mathbb{Q}}_T(\exists t \leq T : |\xi_t - T^q f(t/T)| > \varepsilon T^q) = \tilde{\mathbb{E}}[\zeta_T(T)\mathbb{1}_{\{\exists t \leq T: |\xi_t - T^q f(t/T)| > \varepsilon T^q\}}] = 0$$

by the definition of $\zeta_T(T)$. □

Another important tool is the spine decomposition.

Theorem 4.5 (Spine decomposition)**:**
$\tilde{\mathbb{Q}}_T$-almost surely,

$$\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] = \sum_{u<\xi_t} A_u V_T(S_u) e^{-m\beta\int_0^{S_u}|\xi_s|^p ds} + V_T(t)e^{-m\beta\int_0^t |\xi_s|^p ds}$$

where $\{u < \xi_t\}$ is the set of ancestors of the spine particle at time t, and S_u denotes the time at which particle u died and split into two new particles.

A proof of a more general version of the spine decomposition may be found in [12].

Lemma 4.6:
If $f \in C^2[0,1]$ then for any $u \in N_T(t)$, almost surely under both $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}_T$ we have

$$\left| T^{q-1} \int_0^t f'(s/T) dX_u(s) - T^{2q-1} \int_0^{t/T} f'(s)^2 ds \right| \\ \leq 2\varepsilon T^{2q-1} \int_0^{t/T} |f''(s)| ds + 2\varepsilon T^{2q-1} |f'(0)|.$$

Proof. From the integration by parts formula for Itô calculus (since for any particle $u \in N(t)$, $(X_u(s), 0 \leq s \leq t)$ is a Brownian motion under $\tilde{\mathbb{P}}$) we know that for any $g \in C^2[0,1]$ with $g(0) = 0$, under $\tilde{\mathbb{P}}$,

$$g'(t)X_u(t) = \int_0^t g''(s)X_u(s)ds + \int_0^t g'(s)dX_u(s).$$

From ordinary integration by parts,

$$\int_0^t g'(s)^2 ds = g'(t)g(t) - \int_0^t g(s)g''(s)ds.$$

Now set $g(t) = T^q f(t/T)$ for $t \in [0,T]$. We note that if $u \in N_T(t)$ then $|X_u(s) - g(s)| < \varepsilon T$ for all $s \leq t$. Thus

$$\begin{aligned}
&\left| T^{q-1} \int_0^t f'(s/T) dX_u(s) - T^{2q-1} \int_0^t f'(s)^2 ds \right| \\
&= \left| \int_0^t g'(s) dX_u(s) - \int_0^t g'(s)^2 ds \right| \\
&\leq \left| g'(t)(X_u(t) - g(t)) - \int_0^t g''(s)(X_u(s) - g(s)) ds \right| \\
&\leq 2\varepsilon T \int_0^t |g''(s)| ds + 2\varepsilon T |g'(0)| \\
&= 2\varepsilon T^{2q-1} \int_0^{t/T} |f''(s)| ds + 2\varepsilon T^{2q-1} |f'(0)|
\end{aligned}$$

almost surely under $\tilde{\mathbb{P}}$ and, since $\tilde{\mathbb{Q}}_T \ll \tilde{\mathbb{P}}$ (on $\tilde{\mathcal{F}}_T$), almost surely under $\tilde{\mathbb{Q}}_T$. $\square$

Lemma 4.7:

For any $u \in N_T(t)$,

$$T^{2q-1} \inf_{g \in B(f,\varepsilon)} \int_0^{t/T} |g(s)|^p ds \leq \int_0^t |X_u(s)|^p ds \leq T^{2q-1} \sup_{g \in B(f,\varepsilon)} \int_0^{t/T} |g(s)|^p ds.$$

Proof. This follows immediately from the fact that if $u \in N_T(t)$ then (by definition) there exists $g \in B(f,\varepsilon)$ such that $X_u(s) = T^q g(s/T)$ for all $s \leq t$. □

Lemma 4.8:
If $f \in C^2[0,1]$, $f'(0) = 0$ *and* $m\beta \int_0^\phi |f(s)|^p ds > \frac{1}{2}\int_0^\phi f'(s)^2 ds$ *for all* $\phi \in (0,\theta]$, *then for small enough* $\varepsilon > 0$ *and any* $T > 0$ *and* $t \leq \theta T$, *there exists* $\eta > 0$ *such that*

$$\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] \leq \sum_{u<\xi_t} A_u e^{\pi^2/8\varepsilon^2 T^{2q-1} - \eta \int_0^{S_u} |\xi_s|^p ds} + e^{\pi^2/8\varepsilon^2 T^{2q-1} - \eta \int_0^t |\xi_s|^p ds}$$

$\tilde{\mathbb{Q}}_T$*-almost surely.*

Proof. Recall that under $\tilde{\mathbb{Q}}_T$ the spine is in $N_T(t)$ for all $t \leq T$. Thus by Lemmas 4.6 and 4.7, since $f'(0) = 0$, for any $\eta > 0$

$$\begin{aligned}
&- m\beta \int_0^t |\xi_s|^p ds + T^{q-1} \int_0^t f'(s/T) d\xi_s - \frac{1}{2} T^{2q-1} \int_0^{t/T} f'(s)^2 ds \\
&\qquad \leq -\eta m\beta \int_0^t |\xi_s|^p ds - (1-\eta) m\beta T^{2q-1} \inf_{g \in B(f,\varepsilon)} \int_0^{t/T} |g(s)|^p ds \\
&\qquad\qquad + \frac{1}{2} T^{2q-1} \int_0^{t/T} f'(s)^2 ds + 2\varepsilon T^{2q-1} \int_0^{t/T} |f''(s)| ds
\end{aligned}$$

for all $t \leq T$. Then, since $m\beta \int_0^\phi |f(s)|^p ds > \frac{1}{2}\int_0^\phi f'(s)^2 ds$ for all $\phi \in (0,\theta]$, for small $\varepsilon > 0$ we may choose $\eta > 0$ such that

$$\begin{aligned}
&- (1-\eta) m\beta T^{2q-1} \inf_{g \in B(f,\varepsilon)} \int_0^{t/T} |g(s)|^p ds \\
&\qquad + \frac{1}{2} T^{2q-1} \int_0^{t/T} f'(s)^2 ds + 2\varepsilon T^{2q-1} \int_0^{t/T} |f''(s)| ds \leq 0
\end{aligned}$$

for all $t \in [0, \theta T]$. Plugging this into the spine decomposition, we get

$$\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] \leq \sum_{u<\xi_t} A_u e^{\pi^2/8\varepsilon^2 T^{2q-1} - \eta \int_0^{S_u} |\xi_s|^p ds} + e^{\pi^2/8\varepsilon^2 T^{2q-1} - \eta \int_0^t |\xi_s|^p ds}.$$ □

Proposition 4.9:
If $f \in C^2[0,1]$, $f'(0) = 0$ *and* $m\beta \int_0^\phi |f(s)|^p ds > \frac{1}{2}\int_0^\phi f'(s)^2 ds$ *for all* $\phi \in (0,\theta]$, *then for*

small enough $\varepsilon > 0$ the set $\{Z_T(t) : T \geq 1,\ t \leq \theta T\}$ is uniformly integrable under $\mathbb{P}$.

Proof. Fix $\delta > 0$. We first claim that there exists K such that

$$\sup_{\substack{T\geq 1\\ t\leq\theta T}} \tilde{\mathbb{Q}}_T(\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] > K) < \delta/2.$$

To see this, take an auxiliary probability space with probability measure Q, and on this space consider a sequence $A_1, A_2, \ldots$ of random variables with the same (size-biased) distribution as A under $\tilde{\mathbb{Q}}_T$ (there is no dependence on T) and a sequence $e_1, e_2, \ldots$ of random variables that are exponentially distributed with parameter $\beta(m+1)$; then set $S_n = e_1 + \ldots + e_n$ (so that the random variable S_n has the same distribution as $\int_0^{S_u} |\xi_s|^p ds$, where S_u is the time of the nth fission event along the spine under $\tilde{\mathbb{Q}}_T$). By Lemma 4.8 we have (since $2q - 1 \geq 1$)

$$\sup_{\substack{T\geq 1\\ t\in[1,\theta T]}} \tilde{\mathbb{Q}}_T(\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] > K) \leq Q\left(\sum_{j=1}^{\infty} A_j e^{\pi^2/8\varepsilon^2 - \eta S_j} + e^{\pi^2/8\varepsilon^2} > K\right).$$

Hence our claim holds if the random variable

$$\sum_{j=1}^{\infty} A_j e^{-\eta S_j}$$

can be shown to be Q-almost surely finite. Now for any $\gamma \in (0,1)$,

$$\begin{aligned} Q(\sum_n A_n e^{-\eta S_n} = \infty) &\leq Q(A_n e^{-\eta S_n} > \gamma^n \text{ infinitely often}) \\ &\leq Q\left(\frac{\log A_n}{n} > \log\gamma + \frac{\eta S_n}{n} \text{ infinitely often}\right). \end{aligned}$$

By the strong law of large numbers, $S_n/n \to 1/\beta(m+1)$ almost surely under Q; so if $\gamma \in (\exp(-\eta/\beta(m+1)), 1)$ then the quantity above is no larger than

$$Q\left(\limsup_{n\to\infty} \frac{\log A_n}{n} > 0\right).$$

But this quantity is zero by Borel-Cantelli: for any T,

$$\begin{aligned}\sum_n Q\left(\frac{\log A_n}{n} > \varepsilon\right) &= \sum_n \tilde{\mathbb{Q}}_T(\log A > \varepsilon n)\\ &\le \int_0^\infty \tilde{\mathbb{Q}}_T(\log A \ge \varepsilon x)dx\\ &= \tilde{\mathbb{Q}}_T\left[\frac{\log A}{\varepsilon}\right],\end{aligned}$$

which is finite for any $\varepsilon > 0$ since (by direct calculation from the distribution of A under $\tilde{\mathbb{Q}}_T$ given in Lemma 4.4) $\tilde{\mathbb{Q}}_T[\log A] = \tilde{\mathbb{P}}[A \log A] < \infty$. Thus our claim holds.

Now choose $M > 0$ such that $1/M < \delta/2$; then for K chosen as above, and any $T \ge 1$, $t \le \theta T$,

$$\begin{aligned}\tilde{\mathbb{Q}}_T(Z_T(t) > MK) &\le \tilde{\mathbb{Q}}_T(Z_T(t) > MK,\ \tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] \le K)\\ &\qquad\qquad\qquad + \tilde{\mathbb{Q}}_T(\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T] > K)\\ &\le \tilde{\mathbb{Q}}_T\left[\frac{Z_T(t)}{MK}\mathbb{1}_{\{\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T]\le K\}}\right] + \delta/2\\ &= \tilde{\mathbb{Q}}_T\left[\frac{\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T]}{MK}\mathbb{1}_{\{\tilde{\mathbb{Q}}_T[Z_T(t)|\tilde{\mathcal{G}}_T]\le K\}}\right] + \delta/2\\ &\le 1/M + \delta/2 \le \delta.\end{aligned}$$

Thus, setting $K' = MK$, for any $T \ge 1$, $t \le \theta T$,

$$\mathbb{P}[Z_T(t)\mathbb{1}_{\{Z_T(t)>K'\}}] = \tilde{\mathbb{Q}}_T(Z_T(t) > K') \le \delta.$$

Since $\delta > 0$ was arbitrary, the proof is complete. □

Lemma 4.10:
For any $\delta > 0$, if $f \in C^2[0,1]$, $f(0) = 0$ and ε is small enough then

$$Z_T(\theta T) \le |N_T(f,\varepsilon,\theta)| e^{\frac{\pi^2\theta}{8\varepsilon^2 T^{2q}} - m\beta T^{2q-1}\int_0^\theta |f(s)|^p ds + \frac{1}{2}T^{2q-1}\int_0^\theta f'(s)^2 ds + \delta T^{2q-1}}.$$

Proof. Simply plugging the results of Lemmas 4.6 and 4.7 into the definition of $Z_T(\theta T)$ gives the desired inequality. □

We note here that, in fact, a similar bound can be given in the opposite direction, so that $|N_T(f,\varepsilon/2,\theta)|$ is dominated by $Z^T(\theta T)$ multiplied by some deterministic function of T. We will not need this bound, but it is interesting to note that the study of the martingales Z_T is in a sense equivalent to the study of the number of particles N_T.

4.3 The lower bound

4.3.1 The heuristic for the lower bound

We want to show that $N_T(f,\varepsilon,\theta)$ cannot be too small for large T. For $f\in C[0,1]$ and $\theta\in[0,1]$, define

$$J(f,\theta):=\begin{cases} m\beta\int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds & \text{if } f\in H_1\\ -\infty & \text{otherwise.}\end{cases}$$

We note that J resembles our rate function K, but without the truncation at the extinction time θ_0. We shall work mostly with the simpler object J, before deducing our result involving K at the very last step.

Step 1. Consider a small time ηT. How many particles are in $N_T(f,\varepsilon,\eta)$? If η is much smaller than ε, then (with high probability) no particle has had enough time to reach anywhere near the edge of the tube (approximately distance εT from the origin) before time ηT. Thus, with high probability,

$$|N_T(f,\varepsilon,\eta)| = |N(\eta T)|.$$

We can then give a very simple (and inaccurate!) estimate to show that for some $\nu>0$, with high probability,

$$|N(\eta T)|\geq \nu T.$$

Step 2. Given their positions at time ηT, the particles in $N_T(f,\varepsilon,\eta)$ act independently. Each particle u in this set thus draws out an independent branching Brownian motion. Let $N_T(u,f,\varepsilon,\theta)$ be the set of descendants of u that are in $N_T(f,\varepsilon,\theta)$. How big is this set? Since η is very small, u is close to the origin at time ηT. Thus we may hope to find some $\gamma<1$ such that (for each u)

$$\mathbb{P}\left(|N_T(u,f,\varepsilon,\theta)| < \exp(J(f,\theta)T^{2q-1} - \delta T^{2q-1})\right)\leq\gamma.$$

Step 3. If $N_T(f,\varepsilon,\theta)$ is to be small, then each of the sets $N_T(u,f,\varepsilon,\theta)$ for $u\in N_T(f,\varepsilon,\eta)$ must be small. Thus

$$\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < \exp(J(f,\theta)T^{2q-1} - \delta T^{2q-1})\right)\lesssim\gamma^{\nu T},$$

and we may apply Borel-Cantelli to deduce our result along lattice times (that is, times T_j, $j\geq 0$ such that there exists $\tau>0$ with $T_j-T_{j-1}=\tau$ for all $j\geq 1$).

Step 4. We carry out a simple tube-reduction argument to move to continuous time.

The idea here is that if the result were true on lattice times but not in continuous time, the number of particles in $N_T(f,\varepsilon,\theta)$ must fall dramatically at infinitely many non-lattice times. We simply rule out this possibility using standard properties of Brownian motion.

The most difficult part of the proof is Step 2. However, the spine results of Section 4.2 will simplify our task significantly.

4.3.2 The proof of the lower bound

We begin with Step 1 of our heuristic, considering the size of $N_T(f,\varepsilon,\eta)$ for small η. First we will need the following simple lemma.

Lemma 4.11:
For any $\delta > 0$ and $k > 0$,

$$\tilde{\mathbb{P}}\left(\int_0^t \mathbb{1}_{\{\xi_s\in(-\delta,\delta)\}} ds > k\right) \leq 3e^{t/2-k/4\delta}.$$

Proof. We first claim that if we define $h_\delta : \mathbb{R} \to \mathbb{R}$ by

$$h_\delta(x) := \begin{cases} |x| & \text{if } |x| \geq \delta \\ \frac{\delta}{2} + \frac{x^2}{2\delta} & \text{if } |x| < \delta \end{cases}$$

then

$$h_\delta(\xi_t) = \frac{\delta}{2} + \int_0^t h'_\delta(\xi_s)d\xi_s + \frac{1}{2\delta}\int_0^t \mathbb{1}_{\{\xi_s\in(-\delta,\delta)\}} ds.$$

We check, by approximation with C^2 functions, that Itô's formula holds for h_δ. Define a function $g_{\delta,n} \in C^2(\mathbb{R})$ for each $n \in \mathbb{N}$ by setting

$$g''_{\delta,n}(s) = \begin{cases} 0 & \text{if } |x| \geq \delta \\ \frac{n}{\delta}(\delta - |x|) & \text{if } \delta - \frac{1}{n} < |x| < \delta \\ \frac{1}{\delta} & \text{if } |x| < \delta - \frac{1}{n} \end{cases}$$

with $g'_{\delta,n}(0) = 0$, $g_{\delta,n}(0) = \delta/2$. Since $g \in C^2$, Itô's formula tells us that

$$g_{\delta,n}(\xi_t) = g_{\delta,n}(\xi_0) + \int_0^t g'_{\delta,n}(\xi_s)d\xi_s + \frac{1}{2}\int_0^t g''_{\delta,n}(\xi_s)ds.$$

Since $g''_{\delta,n} \to h''_\delta$ Lebesgue-almost everywhere, by bounded convergence

$$\int_0^t g''_{\delta,n}(\xi_s)ds \to \int_0^t h''_\delta(\xi_s)ds \ \ \tilde{\mathbb{P}}\text{-almost surely,}$$

and $g_{\delta,n} \to h_\delta$ uniformly so for each t, $g_{\delta,n}(\xi_t) \to h_\delta(\xi_t)$ $\tilde{\mathbb{P}}$-almost surely. Also, by the Itô isometry

$$\tilde{\mathbb{P}}\left[\left(\int_0^t (g'_{\delta,n}(\xi_s) - h'_\delta(\xi_s))d\xi_s\right)^2\right] = \tilde{\mathbb{P}}\left[\int_0^t (g'_{\delta,n}(\xi_s) - h'_\delta(\xi_s))^2 ds\right];$$

since $g'_{\delta,n} \to h'_\delta$ uniformly, the right hand side above converges to zero, and hence

$$\int_0^t g'_{\delta,n}(\xi_s)d\xi_s \to \int_0^t h'_\delta(\xi_s)d\xi_s \;\; \tilde{\mathbb{P}}\text{-almost surely.}$$

Thus Itô's formula does indeed hold for h_δ, and since

$$\frac{1}{2}\int_0^t h''_\delta(s)ds = \frac{1}{2\delta}\int_0^t \mathbb{1}_{\{\xi_s\in(-\delta,\delta)\}}ds$$

our claim holds. Now recall that under $\tilde{\mathbb{P}}$, the spine's motion is simply a Brownian motion, so

$$\tilde{\mathbb{P}}[e^{-\int_0^t h'_\delta(\xi_s)d\xi_s}] \leq \tilde{\mathbb{P}}[e^{-\int_0^t h'_\delta(\xi_s)d\xi_s - \frac{1}{2}\int_0^t h'_\delta(\xi_s)^2 ds}]e^{t/2} \leq e^{t/2}.$$

Thus

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\int_0^t \mathbb{1}_{\{\xi_s\in(-\delta,\delta)\}}ds > k\right) &= \tilde{\mathbb{P}}\left(h_\delta(\xi_t) - \frac{\delta}{2} - \int_0^t h'_\delta(\xi_s)d\xi_s > \frac{k}{2\delta}\right)\\
&\leq \tilde{\mathbb{P}}\left(|\xi_t| - \int_0^t h'_\delta(\xi_s)d\xi_s > \frac{k}{2\delta}\right)\\
&\leq \tilde{\mathbb{P}}\left(|\xi_t| > \frac{k}{4\delta}\right) + \tilde{\mathbb{P}}\left(-\int_0^t h'_\delta(\xi_s)d\xi_s > \frac{k}{4\delta}\right)\\
&\leq \tilde{\mathbb{P}}\left[e^{|\xi_t|}\right]e^{-k/4\delta} + \tilde{\mathbb{P}}\left[e^{-\int_0^t h'_\delta(\xi_s)d\xi_s}\right]e^{-k/4\delta}\\
&\leq 3e^{t/2-k/4\delta},
\end{aligned}$$

establishing the result. □

Lemma 4.12:
For any continuous f with $f(0) = 0$ and any $\varepsilon > 0$, there exist $\eta > 0$, $\nu > 0$, $k > 0$ and T_1 such that for all $T \geq T_1$,

$$\mathbb{P}(|N_T(f, \varepsilon/2, \eta)| < \nu T) \leq e^{-kT}.$$

Proof. We first show that there exist $\eta > 0$, $k_1 > 0$ and T_1 such that

$$\mathbb{P}(\exists u \in N(\eta T) : u \notin N_T(f, \varepsilon/2, \eta)) \leq e^{-k_1 T} \quad \forall T \geq T_1.$$

Choose η small enough that $\sup_{s\in[0,\eta]}|f(s)| < \varepsilon/4$. Then, using the many-to-one lemma (at $(\star)$) and standard properties of Brownian motion,

$$\begin{aligned}
&\mathbb{P}(\exists u \in N(\eta T) : u \notin N_T(f,\varepsilon/2,\eta)) \\
&= \mathbb{P}(\exists u \in N(\eta T),\ s \leq \eta : |X_u(sT) - T^q f(s)| \geq \varepsilon T^q/2) \\
&\leq \mathbb{P}(\exists u \in N(\eta T) : \sup_{s\leq \eta T} |X_u(s)| \geq \varepsilon T^q/4) \\
&\leq \sum_{k\geq 1} \mathbb{P}(\exists u \in N(\eta T) : \sup_{s\leq \eta T} |X_u(s)| \in [k\varepsilon T^q/4, (k+1)\varepsilon T^q/4]) \\
&\leq \sum_{k\geq 1} e^{m\beta\int_0^{\eta T}((k+1)\varepsilon T^q)^p ds} \mathbb{P}(\sup_{s\leq \eta T}|\xi_s| \in [k\varepsilon T^q/4, (k+1)\varepsilon T^q/4]) \qquad (\star) \\
&\leq \sum_{k\geq 1} 4e^{m\beta(k+1)^p\varepsilon^p\eta T^{qp+1}} \mathbb{P}(\xi_{\eta T} \in [k\varepsilon T^q/4, (k+1)\varepsilon T^q/4]) \\
&\leq \sum_{k\geq 1} \frac{4}{\sqrt{2\pi\eta T}} \exp\left(m\beta(k+1)^p\varepsilon^p\eta T^{qp+1} - \frac{(k\varepsilon T^q)^2}{32\eta T}\right) \\
&\leq \sum_{k\geq 1} \frac{4}{\sqrt{2\pi\eta T}} \exp\left((m\beta\varepsilon^p\eta - \varepsilon^2/32\eta)kT^{2q-1}\right)
\end{aligned}$$

for sufficiently small η. For small η this is approximately

$$\exp\left((m\beta\varepsilon^p\eta - \varepsilon^2/32\eta)T^{2q-1}\right),$$

which gives the decay required. We now aim to show that there for any $\eta > 0$, there exist $\nu > 0$ and $k_2 > 0$ such that

$$\mathbb{P}(N(\eta T) < \nu T) \leq e^{-k_2 T}.$$

Indeed, if we let $n(t)$ be the number of births along the spine by time t, then certainly

$$\begin{aligned}
&\mathbb{P}(N(\eta T) < \nu T) \\
&\leq \mathbb{P}(n(\eta T) < \nu T) \\
&\leq \mathbb{P}\left(\int_0^{\eta T} \mathbb{1}_{\{\xi_s\in[-(4\nu/\beta\eta)^{1/p},(4\nu/\beta\eta)^{1/p}]\}} ds \geq \frac{1}{2}\eta T\right) \\
&\quad + \mathbb{P}\left(\int_0^{\eta T} \mathbb{1}_{\{\xi_s\in[-(4\nu/\beta\eta)^{1/p},(4\nu/\beta\eta)^{1/p}]\}} ds < \frac{1}{2}\eta T,\ n(\eta T) < \nu T\right).
\end{aligned}$$

Lemma 4.11 shows that

$$\mathbb{P}\left(\int_0^{\eta T} \mathbb{1}_{\{\xi_s\in[-(4\nu/\beta\eta)^{1/p},(4\nu/\beta\eta)^{1/p}]\}} ds \geq \frac{1}{2}\eta T\right) \leq 3\exp\left(\frac{\eta T}{2} - \frac{\eta T}{8(4\nu/\beta\eta)^{1/p})}\right)$$

so we have exponential decay in the first term provided that $\nu < \beta\eta/4^{p+1}$; and since births along the spine occur at rate at least $4\nu/\eta$ outside the interval $[-(4\nu/\beta\eta)^{1/p}, (4\nu/\beta\eta)^{1/p}]$ the second term is bounded above by the probability that a Poisson random variable with mean $2\nu T$ is less than νT. Let $Y \sim \mathrm{Po}(2\nu T)$; then

$$\mathbb{1}_{Y \leq \nu T} = \mathbb{1}_{\exp(\nu T) \geq \exp(Y)} \leq \frac{e^{\nu T}}{e^{Y}}$$

so

$$P(Y \leq \nu T) \leq e^{\nu T} E[e^{-Y}] = e^{\nu T + 2\nu T(\exp(-1)-1)}$$

and this exponent is negative, so the second term also decays exponentially. Finally,

$$\mathbb{P}(|N_T(f,\varepsilon/2,\eta)| < \nu T) \leq \mathbb{P}(\exists u \in N(\eta T) : u \notin N_T(f,\varepsilon/2,\eta)) + \mathbb{P}(N(\eta T) < \nu T)$$

and the proof is complete. □

We now move on to Step 2, using the results of Section 4.2 to bound the probability that we have a small number of particles strictly below 1. The bound given is extremely crude, and there is much room for manoeuvre in the proof, but any improvement would only add unnecessary detail.

Lemma 4.13:
If $f \in C^2[0,1]$ and $J(f,s) > 0$ $\forall s \in (0,\theta]$, then for any $\varepsilon > 0$ and $\delta > 0$ there exists $T_0 \geq 0$ and $\gamma < 1$ such that

$$\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1} - \delta T^{2q-1}}\right) \leq \gamma \quad \forall T \geq T_0.$$

Proof. Note that by Lemma 4.10 for small enough $\varepsilon > 0$ and large enough T,

$$|N_T(f,\varepsilon,\theta)| e^{-J(f,\theta)T^{2q-1} + \delta T^{2q-1}/2} \geq Z_T(\theta T)$$

and hence

$$\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1} - \delta T^{2q-1}}\right) \leq \mathbb{P}\left(Z_T(\theta T) < e^{-\delta T^{2q-1}/2}\right).$$

Suppose first that $f'(0) = 0$. Then $\mathbb{E}[Z_T(\theta T)] = 1$ and, again for small enough ε, by Proposition 4.9 the set $\{Z_T(\theta T), T \geq 1, t \in [1, \theta T]\}$ is uniformly integrable. Thus we may choose K such that

$$\sup_{T \geq 1} \mathbb{E}[Z_T(\theta T)\mathbb{1}_{\{Z_T(\theta T) > K\}}] \leq 1/4,$$

and then

$$\begin{aligned}1 = \mathbb{E}[Z_T(\theta T)] &= \mathbb{E}[Z_T(\theta T)\mathbb{1}_{\{Z_T(\theta T)\le 1/2\}}] + \mathbb{E}[Z_T(\theta T)\mathbb{1}_{\{1/2<Z_T(\theta T)\le K\}}] \\ &\quad + \mathbb{E}[Z_T(\theta T)\mathbb{1}_{\{Z_T(\theta T)>K\}}] \\ &\le 1/2 + K\mathbb{P}(Z_T(\theta T) > 1/2) + 1/4\end{aligned}$$

so that

$$\mathbb{P}(Z_T(\theta T) > 1/2) \ge 1/4K.$$

Hence for large enough T,

$$\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T-\delta T}\right) \le 1 - 1/4K.$$

This is true for all small $\varepsilon > 0$; but increasing ε only increases $|N_T(f,\varepsilon,\theta)|$ so the statement holds for all $\varepsilon > 0$. Finally, if $f'(0) \ne 0$ then choose $g \in C^2[0,\theta]$ such that $g(0) = g'(0) = 0$, $\sup_{s\le\theta}|f-g| \le \varepsilon/2$, $J(g,\phi) > 0\ \forall\phi \le \theta$ and $J(g,\theta) > J(f,\theta) - \delta/2$ (for small η, the function

$$g(t) := \begin{cases} f(t) + at + bt^2 + ct^3 + dt^4 & \text{if } t \in [0,\eta) \\ f(t) & \text{if } t \in [\eta,1] \end{cases}$$

will work for suitable $a,b,c,d \in \mathbb{R}$). Then

$$\begin{aligned}\mathbb{P}(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1}-\delta T^{2q-1}}) &\le \mathbb{P}(|N_T(g,\varepsilon/2,\theta)| < e^{J(g,\theta)T^{2q-1}-\delta T^{2q-1}/2}) \\ &\le 1 - 1/4K\end{aligned}$$

as required. $\square$

We are now ready to carry out step 3 of the heuristic.

Proposition 4.14:
Suppose that $f \in C^2[0,1]$ and $J(f,s) > 0\ \forall s \in (0,\theta]$. Then for lattice times T_j,

$$\liminf_{j\to\infty} \frac{1}{T_j^{2q-1}} \log|N_{T_j}(f,\varepsilon,\theta)| \ge J(f,\theta)$$

almost surely.

Proof. For a particle u, define

$$N_T(u,f,\varepsilon,\theta) := \{v \in N(\theta T) : u \le v,\ \ |X_v(t) - T^q f(t/T)| < \varepsilon T^q\ \ \forall t \in [0,\theta T]\},$$

the set of descendants of u that are in $N_t(f,\varepsilon,\theta)$. Then for $\delta > 0$ and $\eta \in [0,\theta]$,

$$\begin{aligned}
&\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1}-\delta T^{2q-1}} \,\middle|\, \mathcal{F}_{\eta T}\right) \\
&\leq \prod_{u\in N_T(f,\varepsilon/2,\eta)} \mathbb{P}\left(|N_T(u,f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1}-\delta T^{2q-1}} \,\middle|\, \mathcal{F}_{\eta T}\right) \\
&\leq \prod_{u\in N_T(f,\varepsilon/2,\eta)} \mathbb{P}\left(|N_T(g,\varepsilon/2,\theta-\eta)| < e^{J(f,\theta)T^{2q-1}-\delta T^{2q-1}} \right)
\end{aligned}$$

since, given $\mathcal{F}_{\eta T}$, $\{|N_T(u,f,\varepsilon,\theta)| : u \in N_T(f,\varepsilon/2,\eta)\}$ are independent random variables, and where $g : [0,1] \to \mathbb{R}$ is any twice continuously differentiable extension of the function

$$\begin{array}{rccl}
\bar{g}: & [0,\theta-\eta] & \to & \mathbb{R} \\
 & t & \to & f(t+\eta) - f(\eta).
\end{array}$$

If η is small enough, then

$$|J(f,\theta) - J(g,\theta-\eta)| < \delta/2$$

and

$$J(g,s) > 0 \quad \forall s \in (0,\theta-\eta].$$

Hence, applying Lemma 4.13, there exists $\gamma < 1$ such that for all large T,

$$\begin{aligned}
&\mathbb{P}\left(|N_T(g,\varepsilon/2,\theta-\eta)| < e^{J(f,\theta-\eta)T^{2q-1}-\delta T^{2q-1}} \right) \\
&\leq \mathbb{P}\left(|N_T(g,\varepsilon/2,\theta-\eta)| < e^{J(g,\theta-\eta)T^{2q-1}-\delta T^{2q-1}/2} \right) \\
&\leq \gamma.
\end{aligned}$$

Thus for large T,

$$\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1}-\delta T^{2q-1}} \,\middle|\, \mathcal{F}_{\eta T}\right) \leq \gamma^{|N_T(f,\varepsilon/2,\eta)|}. \tag{4.1}$$

Taking expectations in (4.1), and then applying Lemma 4.12, for small η and some $\nu, k > 0$, for large T we have

$$\begin{aligned}
&\mathbb{P}\left(|N_T(f,\varepsilon,\theta)| < e^{J(f,\theta)T^{2q-1}-\delta T^{2q-1}} \right) \\
&\leq \mathbb{P}\left(|N_T(f,\varepsilon/2,\eta)| < \nu T \right) + \gamma^{\nu T} \\
&\leq e^{-kT} + \gamma^{\nu T}.
\end{aligned}$$

The Borel-Cantelli lemma now tells us that for any lattice times T_j, $j \geq 0$,

$$\mathbb{P}\left(\liminf_{j\to\infty} \frac{1}{T_j^{2q-1}} \log |N_j(f,\varepsilon,\theta)| < J(f,\theta) - \delta\right) = 0,$$

and taking a union over $\delta > 0$ gives the result. □

We now move to continuous time using Step 4 of our heuristic.

Proposition 4.15:
Suppose that $f \in C^2[0,1]$ and $J(f,s) > 0$ $\forall s \in (0,\theta]$. Then

$$\liminf_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(f,\varepsilon,\theta)| \geq J(f,\theta)$$

almost surely.

Proof. We claim first that for large enough $j \in \mathbb{N}$, provided that $T_1 \leq 1$,

$$\begin{aligned}
&\left\{ |N_{T_j}(f,\varepsilon,\theta)| > \inf_{t\in[T_j,T_{j+1}]} |N_t(f,2\varepsilon,\theta| \right\} \\
&\quad \subseteq \left\{ \exists v \in N_{T_j}(f,\varepsilon,\theta), u \in N(\theta T_{j+1}) : v \leq u, \sup_{t\in[T_j,T_{j+1}]} |X_u(\theta t) - X_u(\theta T_j)| > \frac{\varepsilon T_j^q}{2} \right\}.
\end{aligned}$$

Indeed, if $v \in N_{T_j}(f,\varepsilon,\theta)$, $t \in [T_j, T_{j+1}]$ and $s \in [0,\theta t]$ then for any descendant u of v at time θt,

$$\begin{aligned}
|X_u(s) - t^q f(s/t)| &\leq |X_u(s) - X_u(s\wedge\theta T_j)| + |X_u(s\wedge\theta T_j) - T_j^q f(\tfrac{s\wedge\theta T_j}{T_j})| \\
&\quad + |T_j^q f(\tfrac{s\wedge\theta T_j}{T_j}) - T_j^q f(s/t)| + |T_j^q f(s/t) - t^q f(s/t)| \\
&\leq |X_u(s) - X_u(s\wedge\theta T_j)| + \varepsilon T_j^q \\
&\quad + T_j^q \sup_{\substack{x,y\in[0,\theta] \\ |x-y|\leq 1/T_j}} |f(x) - f(y)| + \sup_{x\in[0,\theta]} |f(x)||T_{j+1}^q - T_j^q| \\
&\leq |X_u(s) - X_u(s\wedge\theta T_j)| + \frac{3\varepsilon}{2} T_j^q \qquad \text{for large j;}
\end{aligned}$$

so that if any particle is in $N_{T_j}(f,\varepsilon,\theta)$ but does not have a descendant in $N_t(f,2\varepsilon,\theta)$ then its descendants must satisfy

$$\sup_{s\in[\theta T_j,\theta T_{j+1}]} |X_u(s) - X_u(T_j)| \geq \varepsilon T_j^q/2.$$

This is enough to establish the claim, and we deduce via the many-to-one lemma plus

Lemma 4.7 and standard properties of Brownian motion that

$$
\begin{aligned}
&\mathbb{P}\left(|N_{T_j}(f,\varepsilon,\theta)| > \inf_{t\in[T_j,T_{j+1}]} |N_t(f,2\varepsilon,\theta)|\right)\\
&\leq \mathbb{P}\left(\exists v \in N_{T_j}(f,\varepsilon,\theta), u \in N(\theta T_{j+1}) : v \leq u, \sup_{t\in[T_j,T_{j+1}]} |X_u(\theta t) - X_u(\theta T_j)| \geq \varepsilon T_j^q/2\right)\\
&\leq \mathbb{E}\left[\sum_{u\in N(\theta T_{j+1})} \mathbb{1}_{\{|X_u(s)-T_j^q f(s/T_j)|<\varepsilon T_j^q \ \forall s\leq \theta T_j\}} \mathbb{1}_{\{\sup_{t\in[T_j,T_{j+1}]} |X_u(\theta t)-X_u(\theta T_j)|\geq \varepsilon T_j^q/2\}}\right]\\
&= \tilde{\mathbb{E}}\left[e^{m\beta\int_0^{\theta T_{j+1}} |\xi_s|^p ds} \mathbb{1}_{\{\xi_{\theta T_j}\in N_{T_j}(f,\varepsilon,\theta)\}} \mathbb{1}_{\{\sup_{t\in[T_j,T_{j+1}]} |\xi_{\theta t}-\xi_{\theta T_j}|\geq \varepsilon T_j^q/2\}}\right]\\
&\leq e^{m\beta T_j^{2q-1}\sup_{g\in B(f,\varepsilon)}\int_0^\theta |g(s)|^p ds} \tilde{\mathbb{E}}\left[e^{m\beta\int_{\theta T_j}^{\theta T_{j+1}} |\xi_s|^p ds} \mathbb{1}_{\{\sup_{t\in[T_j,T_{j+1}]} |\xi_{\theta t}-\xi_{\theta T_j}|\geq \varepsilon T_j^q/2\}}\right]\\
&\leq e^{m\beta T_j^{2q-1}\sup_{g\in B(f,\varepsilon)}\int_0^\theta |g(s)|^p ds}\\
&\qquad\qquad \cdot \sum_{k=1}^\infty \tilde{\mathbb{E}}\left[e^{m\beta\int_{\theta T_j}^{\theta T_{j+1}} |\xi_s|^p ds} \mathbb{1}_{\{\sup_{t\in[T_j,T_{j+1}]} |\xi_{\theta t}-\xi_{\theta T_j}|\in[k\varepsilon T_j^q/2,(k+1)\varepsilon T_j^q/2]\}}\right]\\
&\leq e^{m\beta T_j^{2q-1}\sup_{g\in B(f,\varepsilon)}\int_0^\theta |g(s)|^p ds + m\beta T_j^{2q-2}(|f(\theta)|+(k+3)\varepsilon/2)}\\
&\qquad\qquad \cdot \sum_{k=1}^\infty \mathbb{P}\Big(\sup_{t\in[0,\theta T_1]} |\xi_t| \in [k\varepsilon T_j^q/2,(k+1)\varepsilon T_j^q/2]\Big)\\
&\leq e^{m\beta T_j^{2q-1}\sup_{g\in B(f,\varepsilon)}\int_0^\theta |g(s)|^p ds + m\beta T_j^{2q-2}(|f(\theta)|+(k+3)\varepsilon/2)}\\
&\qquad\qquad \cdot \sum_{k=1}^\infty \mathbb{P}(\xi_{\theta T_1} \in [k\varepsilon T_j^q/2,(k+1)\varepsilon T_j^q/2])\\
&\leq 4e^{m\beta T_j^{2q-1}\sup_{g\in B(f,\varepsilon)}\int_0^\theta |g(s)|^p ds + m\beta T_j^{2q-2}(|f(\theta)|+(k+3)\varepsilon/2)} \sum_{k=1}^\infty e^{-(k\varepsilon T_j^q)^2/8\theta T_1}
\end{aligned}
$$

which, as in Lemma 4.12, is exponentially small in T_j. Thus the probabilities are summable and we may apply Borel-Cantelli to see that

$$\mathbb{P}(|N_{T_j}(f,\varepsilon,\theta)| > \inf_{t\in[T_j,T_{j+1}]} |N_t(f,2\varepsilon,\theta)| \text{ infinitely often}) = 0.$$

Now,

$$\mathbb{P}\left(\liminf_{T\to\infty}\frac{1}{T^{2q-1}}\log|N_T(f,\varepsilon,\theta)|<J(f,\theta)\right)$$
$$\leq \mathbb{P}\left(\liminf_{j\to\infty}\frac{1}{T_j^{2q-1}}\log|N_{T_j}(f,2\varepsilon,\theta)|<J(f,\theta)\right)$$
$$+\mathbb{P}\left(\liminf_{j\to\infty}\frac{\inf_{t\in[T_j,T_{j+1}]}|N_t(f,\varepsilon,\theta)|}{|N_{T_j}(f,2\varepsilon,\theta)|}<1\right)$$

which is zero by Proposition 4.14 and the above. □

Corollary 4.16:
For any open set $A\subseteq C[0,1]$ and $\theta\in[0,1]$, we have

$$\liminf_{T\to\infty}\frac{1}{T^{2q-1}}\log|N_T(A,\theta)|\geq\sup_{f\in A}K(f,\theta)$$

almost surely.

Proof. Clearly if $\sup_{f\in A}K(f,\theta)=-\infty$ then there is nothing to prove. Thus it suffices to consider the case when there exists $f\in A$ such that $f\in H_1$ and $\theta\leq\theta_0(f)$. Since A is open, in this case we can in fact find $f\in A$ such that $J(f,s)>0$ $\forall s\in(0,\theta]$ (if $J(f,\phi)=0$ for some $\phi\leq\theta$, just choose η small enough that $(1-\eta)f\in A$) and such that f is twice continuously differentiable on $[0,1]$ (the twice continuously differentiable functions are dense in $C[0,1]$). Thus necessarily $\sup_{f\in A}K(f,\theta)>0$, and for any $\delta>0$ we may further assume that $J(f,\theta)>\sup_{f\in A}K(f,\theta)-\delta$. Again since A is open, we may take ε such that $B(f,\varepsilon)\subseteq A$; then clearly for any T

$$N_T(f,\varepsilon,\theta)\subseteq N_T(A,\theta)$$

so by Proposition 4.14 we have

$$\liminf_{T\to\infty}\frac{1}{T^{2q-1}}\log N_T(A,\theta)\geq\sup_{f\in A}K(f,\theta)-\delta$$

almost surely, and by taking a union over $\delta>0$ we may deduce the result. □

4.4 The upper bound

Our plan is as follows: we first rule out the possibility of any particles following unusual paths in Lemma 4.17, which allows us to restrict our attention to a compact set, and hence small balls about sensible paths. We then carry out the task of obtaining a bound along lattice times for balls about such paths in Proposition 4.19. By expanding these balls slightly (using an argument similar to that in Proposition 4.15) we may then bound the growth in continuous time; this is done in Lemma 4.20, and finally we draw this work together in Proposition 4.22 to give the bound in continuous time for any closed set D.

Our first task, then, is to rule out the possibility of any particles following extreme paths. For simplicity of notation, we break with convention by letting

$$\|f\|_\theta := \sup_{s\in[0,\theta]} |f(s)|$$

for $f \in C[0,\theta]$ or $f \in C[0,1]$ (on this latter space, $\|\cdot\|_\theta$ is clearly not a norm, but this will not matter to us). We also extend the definition of $N_T(D,\theta)$ to sets $D \subseteq C[0,\theta]$ in the obvious way, setting

$$N_T(D,\theta) := \{u \in N(\theta T) : \exists f \in D \text{ with } X_u(t) = T^q f(t/T) \;\; \forall t \in [0,\theta T]\}.$$

Lemma 4.17:
Fix $\theta \in [0,1]$. For $N \in \mathbb{N}$, let

$$F_N := \left\{ f \in C[0,\theta] : \exists n \geq N,\; u,s \in [0,\theta] \text{ with } |u-s| \leq \frac{1}{n^2},\; |f(u)-f(s)| > \frac{1}{\sqrt{n}} \right\}.$$

Then for all large N

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(F_N,\theta)| = -\infty$$

almost surely.

Proof. Fix $T \geq S \geq 0$; then for any $t \in [S,T]$,

$$\begin{aligned}\{\xi_t \in N_t(F_N,\theta)\} &= \left\{ \exists n \geq N,\; u,s \in [0,\theta] : |u-s| \leq \frac{1}{n^2},\; \left|\frac{\xi_{ut}-\xi_{st}}{t^q}\right| > \frac{1}{\sqrt{n}} \right\} \\ &\subseteq \left\{ \exists n \geq N,\; u,s \in [0,\theta] : |u-s| \leq \frac{1}{n^2},\; \left|\frac{\xi_{uT}-\xi_{sT}}{S^q}\right| > \frac{1}{\sqrt{n}} \right\}.\end{aligned}$$

Since the right-hand side does not depend on t, we deduce that

$$\begin{aligned}&\{\exists t \in [S,T] : \xi_t \in N_t(F_N,\theta)\} \\ &\qquad \subseteq \left\{ \exists n \geq N,\; u,s \in [0,\theta] : |u-s| \leq \frac{1}{n^2},\; \left|\frac{\xi_{uT}-\xi_{sT}}{S^q}\right| > \frac{1}{\sqrt{n}} \right\}.\end{aligned}$$

Now, for $s \in [0,\theta]$, define $\pi(n,s) := \lfloor 2n^2 s \rfloor / 2n^2$. Suppose we have a continuous function

f such that $\sup_{s\in[0,\theta]} |f(s) - f(\pi(n,s))| \le 1/4\sqrt{n}$. If $u, s \in [0,\theta]$ satisfy $|u-s| \le 1/n^2$, then

$$\begin{aligned}
&|f(u) - f(s)| \\
&\le |f(u) - f(\pi(n,u))| + |f(s) - f(\pi(n,s))| + |f(\pi(n,s)) - f(\pi(n,u))| \\
&\le \frac{1}{4\sqrt{n}} + \frac{1}{4\sqrt{n}} + \frac{2}{4\sqrt{n}} = \frac{1}{\sqrt{n}}.
\end{aligned}$$

Thus

$$\{\exists t \in [S,T] : \xi_t \in N_t(F_N,\theta)\} \subseteq \left\{\exists n \ge N,\ s \le \theta : \left|\frac{\xi_{sT} - \xi_{\pi(n,s)T}}{S^q}\right| > \frac{1}{4\sqrt{n}}\right\}.$$

Standard properties of Brownian motion now give us that

$$\begin{aligned}
\mathbb{P}(\exists t \in [S,T] : \xi_t \in N_t(F_N,\theta)) &\le \mathbb{P}\left(\exists n \ge N,\ s \le \theta : |\xi_{sT} - \xi_{\pi(n,s)T}| > S^q/4\sqrt{n}\right) \\
&\le \sum_{n\ge N} 2n^2 \mathbb{P}\left(\sup_{s\in[0,1/2n^2]} |\xi_{sT}| > S^q/4\sqrt{n}\right) \\
&\le \sum_{n\ge N} 4n^2 \mathbb{P}\left(\sup_{s\in[0,1/2n^2]} \xi_{sT} > S^q/4\sqrt{n}\right) \\
&= \sum_{n\ge N} 4n^2 \mathbb{P}\left(\left|\xi_{T/2n^2}\right| > S^q/4\sqrt{n}\right) \\
&\le \sum_{n\ge N} \frac{8\sqrt{n^3T}}{S^q\sqrt{\pi}} \exp\left(-\frac{S^{2q}n}{16T}\right).
\end{aligned}$$

Taking $S = j$ and $T = j+1$, we note that for large N,

$$\sum_{n\ge N} \frac{8\sqrt{n^3T}}{S\sqrt{\pi}} \exp\left(-\frac{S^{2q}n}{16T}\right) \le \sum_{n\ge N} \exp\left(-\frac{j^{2q-1}N}{32}\right).$$

Now, for any $M > 0$,

$$
\begin{aligned}
&\mathbb{P}(\sup_{t\in[j,j+1]} |N_t(F_N,\theta)| \geq 1) \\
&\leq \mathbb{E}\left[\sum_{u\in N(j+1)} \mathbb{1}_{\{\exists t\in[j,j+1]: u\in N_t(F_N,\theta)\}}\right] \\
&= \mathbb{E}\left[e^{m\beta\int_0^{j+1}|\xi_s|^p ds}\mathbb{1}_{\{\exists t\in[j,j+1]:\xi_t\in N_t(F_N,\theta)\}}\right] \\
&\leq \mathbb{E}\left[e^{m\beta\int_0^{j+1}|\xi_s|^p ds}\mathbb{1}_{\{\exists t\in[j,j+1]:\xi_t\in N_t(F_N,\theta)\}}\mathbb{1}_{\{\sup_{s\leq j+1}|\xi_s|\leq M(j+1)^q\}}\right] \\
&\qquad\qquad + \mathbb{E}\left[e^{m\beta\int_0^{j+1}|\xi_s|^p ds}\mathbb{1}_{\{\sup_{s\leq j+1}|\xi_s|>M(j+1)^q\}}\right] \\
&\leq e^{m\beta M^p(j+1)^{pq+1}}\mathbb{P}(\exists t\in[j,j+1]:\xi_t\in N_t(F_N,\theta)) \\
&\qquad\qquad + \sum_{k\geq 1}\mathbb{E}\left[e^{m\beta\int_0^{j+1}|\xi_s|^p ds}\mathbb{1}_{\{\sup_{s\leq j+1}|\xi_t|\in[kM(j+1)^q,(k+1)M(j+1)^q]\}}\right] \\
&\leq e^{m\beta M^p(j+1)^{pq+1}}\mathbb{P}(\exists t\in[j,j+1]:\xi_t\in N_t(F_N,\theta)) \\
&\qquad\qquad + \sum_{k\geq 1} e^{m\beta(j+1)^{2q-1}(k+1)^p M^p}\mathbb{P}(\sup_{s\leq j+1}|\xi_s|\in[kM(j+1)^q,(k+1)M(j+1)^q]) \\
&\leq e^{m\beta M^p(j+1)^{pq+1}}\mathbb{P}(\exists t\in[j,j+1]:\xi_t\in N_t(F_N,\theta)) \\
&\qquad\qquad + 4\sum_{k\geq 1}\frac{1}{\sqrt{2\pi}(j+1)}e^{m\beta(j+1)^{2q-1}(k+1)^p M^p - k^2M^2(j+1)^{2q-1}/2}.
\end{aligned}
$$

Both of these terms (the first by our calculations earlier in the proof) can be made exponentially small in j by choosing M, and then N, sufficiently large. Thus by Borel-Cantelli we have that for large enough N

$$\mathbb{P}(\limsup_{j\to\infty}\sup_{t\in[j,j+1]}|N_t(F_N,\theta)|\geq 1) = 0$$

and since $|N_T(F_N,\theta)|$ is integer-valued,

$$\limsup_{T\to\infty}\frac{1}{T^{2q-1}}\log|N_T(F_N,\theta)| = -\infty$$

almost surely. $\square$

We now attempt to establish an upper bound along lattice times for closed balls about functions outside F_N. First we need the following simple lemma.

Lemma 4.18:
For any $x, y \in \mathbb{R}$,

$$|x+y|^p \leq |x|^p + |y|^p + 2|x|^{p/2}|y|^{p/2}.$$

Proof. If $y = 0$ or $p = 0$ then the result is clear; otherwise dividing through by $|y|^p$ we see that it suffices to show that for any $x \in \mathbb{R}$,

$$|x+1|^p \leq |x|^p + 1 + 2|x|^{p/2}.$$

If $x < -1$ then $|x+1|^p \leq |x|^p$, and if $-1 \leq x \leq 0$ then $|x+1|^p \leq 1$, so we need only consider the case $x > 0$. In this case, by dividing through by x^p we see that the desired inequality holds for x if and only if it holds for $1/x$, so it suffices to check the case $x \leq 1$. Consider the function $\Gamma : [0,\infty) \to \mathbb{R}$ given by

$$\Gamma(x) := (1+x)^p - 1 - x^p - 2x^{p/2};$$

then Γ is continuously differentiable, $\Gamma(0) = 0$ and we claim that

$$\Gamma'(x) = p(1+x)^{p-1} - px^{p-1} - px^{p/2-1} \leq 0 \quad \forall x \in (0,1];$$

if this claim holds then we are done. If $p \in (0,1]$ then the claim clearly holds since $1 + x > x$ for all x and $p - 1 \leq 0$. So suppose that $p \in (1,2)$. But for $p \in (1,2)$ and $x > 0$, we have $(1+x)^{p-1} \leq 1 + x^{p-1}$ (we mentioned this result in Theorem 1.7 — it can be checked by differentiating) so for $x \in (0,1]$

$$(1+x)^{p-1} \leq 1 + x^{p-1} \leq x^{p/2-1} + x^{p-1}$$

since $p/2 - 1 < 0$. This establishes the claim and completes the proof of the lemma. $\square$

In a slight abuse of notation, for $D \subseteq C[0,1]$ and $\theta \in [0,1]$ we define

$$J(D,\theta) := \sup_{f \in D} \left\{ m\beta \int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds \right\}.$$

Proposition 4.19:
For any closed ball $D = \overline{B(f,\varepsilon)} \subseteq C[0,1]$ about any $f \notin F_N$, and any $\theta \in [0,1]$ and lattice times T_j, we have

$$\limsup_{j\to\infty} \frac{1}{T_j^{2q-1}} \log |N_{T_j}(D,\theta)| \leq J(D,\theta) + R_N(\varepsilon)$$

almost surely, where

$$R_N(\varepsilon) := \begin{cases} 0 & \text{if } p = 0 \\ 2m\beta \left(\frac{N^2+1}{\sqrt{N}} + \varepsilon\right)^{p/2} (2\varepsilon)^{p/2} + (2\varepsilon)^p & \text{if } p > 0; \end{cases}$$

in particular R is a deterministic function of ε such that for each N, $R_N(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. From the upper bound for Schilder's theorem (Theorem 5.1 of [36]) we have

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log \mathbb{P}(\xi_T \in N_T(D,\theta)) \leq -\inf_{f\in D} \frac{1}{2}\int_0^\theta f'(s)^2 ds.$$

Thus, by the many-to-one lemma,

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log \mathbb{E}\big[|N_T(D,\theta)|\big] &\leq \limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log \mathbb{E}\left[e^{m\beta\int_0^{\theta T}|\xi_s|^p ds}\mathbb{1}_{\{\xi_T\in N_T(D,\theta)\}}\right] \\ &\leq \sup_{g\in D} m\beta \int_0^\theta |g(s)|^p ds - \inf_{g\in D}\frac{1}{2}\int_0^\theta g'(s)^2 ds.\end{aligned}$$

Suppose now that $p > 0$. Note that since $f \notin F_N$, $\sup_{s\in[0,\theta]}|f(s)| \leq (N^2+1)/\sqrt{N}$ (split $[0,\theta]$ into N^2+1 intervals of equal width; then f changes by at most $1/\sqrt{N}$ on each interval). Now fix $\delta > 0$ and choose $h \in D$ such that

$$\int_0^\theta h'(s)^2 ds \leq \inf_{g\in D}\int_0^\theta g'(s)^2 ds + \delta.$$

For any $g \in D$,

$$\begin{aligned}\int_0^\theta |g(s)|^p ds &\leq \int_0^\theta (|h(s)| + 2\varepsilon)^p ds \\ &\leq \int_0^\theta |h(s)|^p ds + 2\int_0^\theta |h(s)|^{p/2}(2\varepsilon)^{p/2} ds + \int_0^\theta (2\varepsilon)^p ds \\ &\leq \int_0^\theta |h(s)|^p ds + 2\left(\frac{N^2+1}{\sqrt{N}} + \varepsilon\right)^{p/2}(2\varepsilon)^{p/2} + (2\varepsilon)^{p/2}.\end{aligned}$$

Thus

$$m\beta\int_0^\theta |h(s)|^p ds - \frac{1}{2}\int_0^\theta h'(s)^2 ds \geq \sup_{g\in D} m\beta\int_0^\theta |g(s)|^p ds - \inf_{g\in D}\frac{1}{2}\int_0^\theta g'(s)^2 ds - \delta - R_N(\varepsilon)$$

where

$$R_N(\varepsilon) := 2m\beta\left(\frac{N^2+1}{\sqrt{N}} + \varepsilon\right)^{p/2}(2\varepsilon)^{p/2} + (2\varepsilon)^p.$$

Since $\delta > 0$ was arbitrary, this entails that

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log \mathbb{E}\big[|N_T(D,\theta)|\big] \leq J(D,\theta) + R_N(\varepsilon)$$

which holds (trivially) also for $p = 0$.

Applying Markov's inequality, for any $\delta > 0$ and $p \in [0, 2)$ we get

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log \mathbb{P}\big(|N_T(D,\theta)| \geq e^{J(D,\theta)T^{2q-1}+R_N(\varepsilon)T^{2q-1}+\delta T^{2q-1}}\big) \leq \limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log \frac{\mathbb{E}\big[|N_T(D,\theta)|\big]}{e^{J(D,\theta)T^{2q-1}+R_N(\varepsilon)T^{2q-1}+\delta T^{2q-1}}} \leq -\delta$$

so that for lattice times $T_1, T_2, \ldots$ we have

$$\sum_{j=1}^{\infty} \mathbb{P}\big(|N_{T_j}(D,\theta)| \geq e^{J(D,\theta)T_j^{2q-1}+R_N(\varepsilon)T_j^{2q-1}+\delta T_j^{2q-1}}\big) < \infty$$

and hence by the Borel-Cantelli lemma

$$\mathbb{P}\left(\limsup_{j\to\infty} \frac{1}{T_j^{2q-1}} \log |N_{T_j}(D,\theta)| \geq J(D,\theta) + R_N(\varepsilon) + \delta\right) = 0.$$

Taking a union over $\delta > 0$ now gives the result. $\square$

We now check that an upper bound holds in continuous time. For $\delta > 0$ and $D \subseteq C[0,1]$, define

$$D^\delta := \{f \in C[0,1] : \exists g \in D \text{ with } \|f - g\| \leq \delta\}.$$

Lemma 4.20:
If $D = \overline{B(f,\varepsilon)} \subseteq C[0,1]$ for some $f \notin F_N$, then

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| \leq J(D^\varepsilon,\theta) + R_N(2\varepsilon)$$

almost surely.

Proof. First note that for lattice times $T_1, T_2, \ldots$,

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| > J(D^\varepsilon,\theta) + R_N(2\varepsilon) + \delta\right) \leq \mathbb{P}\left(\limsup_{j\to\infty} \frac{1}{T_j^{2q-1}} \log |N_{T_j}(D^\varepsilon,\theta)| > J(D^\varepsilon,\theta) + R_N(2\varepsilon)\right) + \mathbb{P}\left(\limsup_{j\to\infty} \frac{1}{T_j^{2q-1}} \log \sup_{t\in[T_j,T_{j+1}]} \frac{|N_t(D,\theta)|}{|N_{T_j}(D^\varepsilon,\theta)|} > \delta\right).$$

Clearly $D^\varepsilon = \overline{B(f,2\varepsilon)}$, so immediately by Proposition 4.19,

$$\mathbb{P}\left(\limsup_{j\to\infty} \frac{1}{T_j^{2q-1}} \log |N_{T_j}(D^\varepsilon,\theta)| > J(D^\varepsilon,\theta) + R_N(2\varepsilon)\right) = 0$$

and we may concentrate on the last term. We claim that for j large enough, provided that $T_1 \leq 1$, for any $t \in [T_j, T_{j+1}]$ we have

$$u \in N_t(D,\theta) \Rightarrow \exists v \leq u \text{ with } v \in N_{T_j}(D^\varepsilon,\theta).$$

Indeed, if $u \in N_t(D,\theta)$ then for any $s \leq \theta T_j$,

$$\begin{aligned} |X_u(s) - T_j^q f(s/T_j)| &\leq |X_u(s) - t^q f\left(s/t\right)| + |T_j^q f\left(s/T_j\right) - t^q f\left(s/T_j\right)| \\ &\qquad\qquad + t^q |f\left(s/T_j\right) - f\left(s/t\right)| \\ &\leq t^q\varepsilon + \|f\|_\theta (T_{j+1}^q - T_j^q) + t^q \sup_{\substack{x,y\in[0,\theta]\\ |x-y|\leq 1/T_j}} |f(x)-f(y)| \end{aligned}$$

which is smaller than $2\varepsilon T_j^q$ for large j since f is absolutely continuous.

We deduce that for large j every particle in $N_t(D,\theta)$ for any $t \in [T_j, T_{j+1}]$ has an ancestor in $N_{T_j}(D^\varepsilon,\theta)$. We now use this fact to ensure that $N_t(D,\theta)$ cannot increase dramatically between times T_j and T_{j+1}.

We temporarily need some more notation. For $t > s \geq 0$ and $u \in N(s)$, let $N(u,s,t)$ be the set of descendants of u born between times s and t. Also let $\tilde{\mathbb{P}}_x$ be the translation of $\tilde{\mathbb{P}}$ under which we start with one particle at x rather than at the origin. Then, using the Markov property and the many-to-one lemma,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[T_j,T_{j+1}]}|N_t(D,\theta)|\,\middle|\,\mathcal{F}_{\theta T_j}\right]\\
&\leq \mathbb{E}\left[\sum_{u\in N_{T_j}(D^\varepsilon,\theta)}|N(u,\theta T_j,\theta T_{j+1})|\,\middle|\,\mathcal{F}_{\theta T_j}\right]\\
&\leq \sum_{u\in N_{T_j}(D^\varepsilon,\theta)}\mathbb{E}_{X_u(\theta T_j)}\left[|N(\theta T_1)|\right]\\
&= \sum_{u\in N_{T_j}(D^\varepsilon,\theta)}\mathbb{E}_{X_u(\theta T_j)}\left[e^{m\beta\int_0^{\theta T_1}|\xi_s|^p ds}\right]\\
&\leq \sum_{u\in N_{T_j}(D^\varepsilon,\theta)}\sum_{k\geq 0}e^{m\beta\theta T_1(|X_u(\theta T_j)|+k+1)^p}\mathbb{P}_{X_u(\theta T_j)}\left(\sup_{s\in[0,\theta T_1]}|\xi_s-\xi_0|\in[k,k+1]\right)\\
&\leq |N_{T_j}(D^\varepsilon,\theta)|\sum_{k\geq 0}e^{m\beta\theta T_1(T_j^q(\|f\|_\theta+2\varepsilon)+k+1)^p}\frac{4e^{-k^2/2\theta T_1}}{\sqrt{2\pi\theta T_1}}
\end{aligned}$$

By choosing T_1 small, we may ensure that this sum converges, giving

$$\mathbb{E}\left[\sup_{t\in[T_j,T_{j+1}]}|N_t(D,\theta)|\,\middle|\,\mathcal{F}_{\theta T_j}\right]\leq |N_{T_j}(D^\varepsilon,\theta)|e^{O(T_j^{pq})}.$$

But $pq=2q-2$ and by Markov's inequality

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[T_j,T_{j+1}]}\frac{|N_t(D,\theta)|}{|N_{T_j}(D^\varepsilon,\theta)|}>\exp\left(\delta T_j^{2q-1}\right)\right)\\
&\leq \mathbb{E}\left[\frac{\mathbb{E}\left[\sup_{t\in[T_j,T_{j+1}]}|N_t(D,\theta)|\,\middle|\,\mathcal{F}_{\theta T_j}\right]}{|N_{T_j}(D^\varepsilon,\theta)|}\right]\exp(-\delta T_j^{2q-1})\\
&\leq \exp(O(T_j^{2q-2})-\delta T_j^{2q-1}).
\end{aligned}$$

Thus we may apply Borel-Cantelli to see that

$$\mathbb{P}\left(\limsup_{j\to\infty}\frac{1}{T_j^{2q-1}}\log\sup_{t\in[T_j,T_{j+1}]}\frac{|N_t(D,\theta)|}{|N_{T_j}(D^\varepsilon,\theta)|}>\delta\right)=0.$$

Again taking a union over $\delta>0$ gives the result. $\square$

We now check that we can cover our sets in a suitable way.

Lemma 4.21:
For $\theta \in [0,1]$, let

$$C_0[0,\theta] := \{f \in C[0,\theta] : f(0) = 0\}.$$

For each $N \in \mathbb{N}$, the set $C_0[0,\theta] \setminus F_N$ is totally bounded under $\|\cdot\|_\theta$ (that is, it may be covered by open balls of arbitrarily small radius).

Proof. Given $\varepsilon > 0$, choose n such that $n \geq N \vee 1/\varepsilon^2$. For any $f \in C_0[0,\theta] \setminus F_N$, if $|u - s| < 1/n^2$ then $|f(u) - f(s)| \leq 1/\sqrt{n} \leq \varepsilon$. Thus $C_0[0,\theta] \setminus F_N$ is equicontinuous (and, since each function must start from 0, uniformly bounded) and we may apply the Arzelà-Ascoli theorem to say that $C_0[0,\theta] \setminus F_N$ is relatively compact, which is equivalent to totally bounded since $(C[0,\theta], \|\cdot\|_\theta)$ is a complete metric space. □

We are now in a position to give an upper bound for any closed set D in continuous time. This upper bound is not quite what we asked for in Theorem 4.1, but the final step — replacing J with K — will be carried out in Corollary 4.23.

Proposition 4.22:
If $D \subset C[0,1]$ is closed, then for any $\theta \in [0,1]$

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| \leq J(D,\theta)$$

almost surely.

Proof. Clearly (since our first particle starts from 0) $N_T(D \backslash C_0[0,1], \theta) = \emptyset$ for all T, so we may assume without loss of generality that $D \subseteq C_0[0,1]$. Now, for each θ, $f \mapsto \int_0^\theta f'(s)^2 ds$ is a lower semicontinuous function on $C_0[0,\theta]$: we refer to Section 5.2 of [6] but it is possible to give a direct proof. Thus $f \mapsto m\beta \int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds$ is clearly upper semicontinuous. Now, by Jensen's inequality, for any $f \in C_0[0,\theta]$ and any $s,t \in [0,\theta]$, $s < t$,

$$\frac{1}{t-s}\int_s^t f'(u)^2 du \geq \left(\frac{1}{t-s}\int_s^t f'(u)du\right)^2 = \left(\frac{f(t)-f(s)}{t-s}\right)^2$$

so that

$$(f(t) - f(s))^2 \leq (t-s)\int_s^t f'(u)^2 du. \tag{4.2}$$

There exists $t \in [0,\theta]$ such that $|f(t)|^p \geq \frac{1}{\theta}\int_0^\theta |f(s)|^p ds$, so by (4.2) (taking $s = 0$)

$$\int_0^\theta f'(u)^2 du \geq \int_0^t f'(u)^2 du \geq \frac{\left(\int_0^\theta |f(s)|^p ds\right)^{2/p}}{\theta^{2/p} t} \geq \left(\int_0^\theta |f(s)|^p ds\right)^{2/p}$$

and hence

$$\begin{aligned}\{f \in C_0[0,\theta] : m\beta \int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds \geq K\}\\ \subseteq \{f \in C_0[0,\theta] : m\beta \left(\int_0^\theta f'(s)^2 ds\right)^{p/2} - \frac{1}{2}\int_0^\theta f'(s)^2 ds \geq K\}\\ \subseteq \{f \in C_0[0,\theta] : \int_0^\theta f'(s)^2 ds \leq K'\}\end{aligned}$$

for some K' since $p/2 < 1$. But by (4.2),

$$\begin{aligned}\{f \in C_0[0,\theta] : \int_0^\theta f'(s)^2 ds \leq K'\}\\ \subseteq \{f \in C_0[0,\theta] : \forall s,t \in [0,\theta], |f(s) - f(t)| \leq \sqrt{(t-s)K'}\}\end{aligned}$$

and the Arzelà-Ascoli theorem tells us that this latter set is totally bounded. Thus the set

$$\{f \in C_0[0,\theta] : m\beta \int_0^\theta |f(s)|^p ds - \frac{1}{2}\int_0^\theta f'(s)^2 ds \geq J(D,\theta) + \delta\}$$

is totally bounded, but by upper-semicontinuity it is closed, and hence compact. Since it is disjoint from $\{f \in C_0[0,\theta] : \exists g \in D \text{ with } f(s) = g(s)\ \forall s \in [0,\theta]\}$, which is closed, there is a positive distance between the two sets. Now fix $\delta > 0$ and choose N (by Lemma 4.17) such that

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(F_N,\theta)| = -\infty;$$

by the above and the fact that $R_N(2\varepsilon) \to 0$ as $\varepsilon \to 0$, we may choose $\varepsilon > 0$ such that $J(D^\varepsilon,\theta) + R_N(2\varepsilon) < J(D,\theta) + \delta$. Then, by Lemma 4.21, for any N and some α (depending on N) and $f_k \in C[0,1] \setminus F_N$, $k = 1, 2, \ldots, \alpha$,

$$\begin{aligned}&\mathbb{P}\left(\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| > J(D,\theta) + \delta\right)\\ &\leq \mathbb{P}\left(\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(F_N,\theta)| > J(D,\theta) + \delta\right)\\ &\qquad + \sum_{k=1}^{\alpha} \mathbb{P}\left(\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(f_k,\varepsilon,\theta)| > J(D^\varepsilon,\theta) + R_N(2\varepsilon)\right).\end{aligned}$$

By our choice of N, the first term on the right-hand side is zero, and by Lemma 4.20 all of the terms in the sum are also zero. As usual we take a union over $\delta > 0$ to complete the proof. $\square$

Corollary 4.23:
For any closed set $D \subseteq C[0,1]$ *and* $\theta \in [0,1]$*, we have*

$$\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| \leq \sup_{f\in D} K(f,\theta)$$

almost surely.

Proof. Since $|N_T(D,\theta)|$ is integer valued,

$$\frac{1}{T^{2q-1}} \log |N_T(D,\theta)| < 0 \quad \Rightarrow \quad \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| = -\infty.$$

Thus, by Proposition 4.19, if $J(D,\theta) < 0$ then

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| > -\infty\right) = 0.$$

Further, clearly for $\phi \leq \theta$ and any $T \geq 0$, if $N_T(D,\phi) = \emptyset$ then necessarily $N_T(D,\theta) = \emptyset$. Thus if there exists $\phi \leq \theta$ with $J(D,\phi) < 0$, then

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{1}{T^{2q-1}} \log |N_T(D,\theta)| > -\infty\right) = 0$$

which completes the proof. □

Combining Corollary 4.16 with Corollary 4.23 completes the proof of Theorem 4.1.

Chapter 5

BBM: Behaviour along unscaled paths

For a set $A \subset C[0,\infty)$, we give new results on the growth of the number of particles in a binary branching Brownian motion whose paths fall within A. We show that it is possible to work without rescaling the paths. We give large deviations probabilities as well as a more sophisticated proof of a result on growth in the number of particles along certain sets of paths. Our results reveal that the number of particles can oscillate dramatically. As a byproduct of our methods, we also obtain new results on the number of particles near the critical frontier of the BBM. The methods used are entirely probabilistic. This chapter makes a significant improvement on the results of Harris and Roberts [16].

5.1 Introduction

The classical *scaled* path properties of branching Brownian motion (BBM) have now been well-studied: for example, see Lee [30] and Hardy and Harris [10] for large deviation results on "difficult" paths which have a small probability of any particle following them, and Git [8] and Harris and Roberts [17], as well as Chapter 4 of this thesis, for the almost sure growth rate of the number of particles near "easy" paths along which we see exponential growth in the number of particles. To give these results, the paths of a BBM are rescaled onto the interval $[0,1]$, echoing the approach of Schilder's theorem for a single Brownian motion.

Here we consider a problem similar in theme, but from a more naive viewpoint. We are given a fixed set of paths $A \subset C[0,\infty)$ and we want to know how many particles in a BBM have paths within this set A. Similar problems in the case of a single Brownian motion have been considered by Kesten [25] and Novikov [34]. The simplest case is to consider the ball $B(f, L)$ of fixed width $L > 0$ about a single continuous path $f : [0, \infty) \to \mathbb{R}$ — and this is covered in Harris and Roberts [16]. Clearly there is a positive probability that no particle will stay within this fixed "tube" (indeed, the very first particle could wander away from f before it has the chance to give birth to another): in this event we say that the process becomes extinct.

The intuition is that the growth of the population due to branching is in constant competition with the "deaths" due to particles failing to follow the function f. Thus a natural condition arises: if the gradient of f is too large, then the process eventually dies out almost surely and we may ask for the large deviation probabilities of survival up to large times; otherwise, if the gradient of f remains sufficiently small, then we may condition on non-extinction and give an almost sure result on the number of particles along the path.

The payoff for our less classical approach is that we immediately see a dramatic oscillation in the number of particles along certain paths. This unusual behaviour (not seen in the existing literature) has a simple explanation which we demonstrate via some illuminating examples in Section 5.3.

We take advantage of spine techniques to interpret the change of measure given by a carefully chosen martingale. The spine tools give us an intuitive probabilistic handle on the problem, without which we would certainly need substantial extra technical work in several areas. Our particular change of measure involves forcing one particle (the spine) to stay within a tube of varying radius $L(t)$, $t \geq 0$ about a function f. This change of measure is the result of a new martingale which we develop in Section 4.2. We then use the spine decomposition first introduced by Lyons *et al.* [32], which allows us to bound the growth of the system by looking at the births along the spine.

Even with the spine theory the problem retains significant difficulty inherent in its time-inhomogeneity. This fact is underlined by the observation that even in the case $A = B(f, L)$ we are essentially considering a one-dimensional branching diffusion with time-dependent drift, and asking how many particles remain within a bounded domain about the origin. It turns out that the main difficulty is in showing that extinction of the process coincides (to within a null set) with the event that the limit of our martingale is zero. Standard tools – analytic or probabilistic – cannot be applied; instead we proceed by our own methods in Section 5.6, using in particular the identity from Lemma 3.4.

For simplicity, we consider only standard one-dimensional binary branching Brownian motion, but we note that our work could be extended to a wide range of other branching diffusions. In particular the spine methods are well-suited to the situation where each particle gives birth to a random number of new particles, and methods similar to those used in the original papers of Lyons *et al.* [27, 31, 32] — and seen in Chapter 4 of this thesis — could be used to extend our result.

Finally, using the same methods as for our main theorem, we are able to obtain new results on the number of particles near the extremes of the system that should be compared with the work of Bramson [4] on the position of the right-most particle, and of Kesten [25] and other authors on BBM with absorption.

5.2 Main results

5.2.1 Initial definitions

We consider a branching Brownian motion starting with one particle at the origin, whereby each particle moves independently and undergoes independent binary branching at exponential rate $r > 0$ — that is, the birth distribution A satisfies $\mathbb{P}(A = 1) = 1$ so each particle gives birth to two children when it dies. We let the set of particles alive at time t be $N(t)$, and for each particle $u \in N(t)$ denote its position at time t by $X_u(t)$. We extend this notion of a particle's position to include the positions of its ancestors; that is, if $u \in N(t)$ has ancestor $v \in N(s)$ for some $s < t$, then we set $X_u(s) := X_v(s)$. This setup was formalised in Chapter 2.

Fix a continuous function $f : [0, \infty) \to \mathbb{R}$, and another $L : [0, \infty) \to (0, \infty)$. If f and L are twice continuously differentiable then we define

$$E(t) := |f'(t)|L(t) + \int_0^t |f''(s)|L(s)ds + \frac{1}{2}|L'(t)|L(t) + \frac{1}{2}\int_0^t |L''(s)|L(s)ds$$

and

$$S := \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)} \right) ds.$$

We say that the pair (f, L) satisfies the *usual conditions* if:

(I) $f(0) = 0$;

(II) f and L are twice continuously differentiable;

(III) $\lim_{t\to\infty} E(t)/t = 0$;

(IV) $S \in (-\infty, \infty)$.

We assume unless otherwise stated that these conditions hold, and consider initially the class of sets of the form

$$B(f, L) := \{g \in C[0, \infty) : |g(t) - f(t)| < L(t) \ \ \forall t \in [0, \infty)\}$$

such that f and L satisfy the usual conditions. After we obtain our results we will be able to extend them in a natural way to cover more general subsets of $C[0,\infty)$ — see Section 5.7 — but for now these conditions will allow us to apply integration by parts theorems without any complications. Although condition (III) may appear unnatural, there are clear reasons behind it, some of which are demonstrated via example in Section 5.7. There are also similar conditions in the work on a single Brownian motion by Kesten [25] and Novikov [34].

Define

$$\hat{N}(t) := \{u \in N(t) : |X_u(s) - f(s)| < L(s) \;\; \forall s \leq t\},$$

the set of particles that have stayed within distance L of the function f for all times $s \leq t$. We wish to study the number of particles in $\hat{N}(t)$ at large times. Let

$$\Upsilon := \inf\{t \geq 0 : \hat{N}(t) = \emptyset\}.$$

We call Υ the *extinction time* for the process, and say that the process has become *extinct* by time t if $\Upsilon \leq t$. When we talk about *survival* or *non-extinction*, we mean the event $\Upsilon = \infty$.

5.2.2 The main result

We now state our main result. Most of this article will be concerned with proving this theorem.

Theorem 5.1:
If $S < 0$, then $\Upsilon < \infty$ almost surely and

$$\frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{\inf_{s\leq t} \int_0^s \left(r - \frac{1}{2}f'(u)^2 - \frac{\pi^2}{8L(u)^2} + \frac{L'(u)}{2L(u)}\right) du} \longrightarrow 1.$$

On the other hand, if $S > 0$, then $\mathbb{P}(\Upsilon = \infty) > 0$ and almost surely on survival we have

$$\frac{\log |\hat{N}(t)|}{\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds} \longrightarrow 1.$$

As mentioned earlier, the theorem can be extended to cover more general sets, and we give results in this direction in Section 5.7. The behaviour at criticality (S=0) remains largely open: it depends on the finer behaviour of f and L, although we are able to give some results in particular cases in Section 5.8. We note the following corollary, which is easily deduced from Theorem 5.1.

Corollary 5.2:
If $S > 0$, then almost surely on survival we have

$$\limsup_{t\to\infty} \frac{1}{t} \log |\hat{N}(t)| = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(r - \frac{\pi^2}{8L(s)^2} - \frac{1}{2} f'(s)^2 + \frac{L'(s)}{2L(s)} \right) ds$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \log |\hat{N}(t)| = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \left(r - \frac{\pi^2}{8L(s)^2} - \frac{1}{2} f'(s)^2 + \frac{L'(s)}{2L(s)} \right) ds.$$

This dramatic oscillation in the number of particles along certain paths at large times is not usually seen in the branching processes literature. Example 5.7 below helps to show why it occurs in our situation.

5.3 Examples

We now consider some very simple examples to give the reader a flavour of the implications of Theorem 5.1. More complex examples will be given in Sections 5.7 and 5.8 in order to explore the limits of our method.

Example 5.3:
Take $f(t) = \lambda t$ with $\lambda \in \mathbb{R}$ and $L(t) \equiv L > 0$. We have a growth rate of $r - \frac{\lambda^2}{2} - \frac{\pi^2}{8L^2}$ (provided this is non-zero): if this constant is negative, then

$$\frac{1}{t} \log \mathbb{P}(\hat{N}(t) \neq \emptyset) \quad \longrightarrow \quad r - \frac{\lambda^2}{2} - \frac{\pi^2}{8L^2}$$

and if it is positive then there is a strictly positive probability of survival, and almost surely on that event

$$\frac{1}{t} \log \hat{N}(t) \quad \longrightarrow \quad r - \frac{\lambda^2}{2} - \frac{\pi^2}{8L^2}$$

Thus taking a fixed L introduces an extra "killing" rate of $\frac{\pi^2}{8L^2}$ to the system compared to the scaled results of Chapter 4 and [8, 10, 17, 30].

Example 5.4:
Again take $f(t) = \lambda t$ with $\lambda \in \mathbb{R} \setminus \{\sqrt{2r}\}$ but now let L be any unbounded monotone non-decreasing function such that (f, L) satisfies the usual conditions (for example $L(t) = (t+1)^\beta$ with $\beta \in (0,1)$ or $L(t) = \log(t+2)$). Then we have a growth rate of $r - \frac{\lambda^2}{2}$: thus while constant L severely restricts the growth of the system, as soon as we relax L slightly we regain the full growth behaviour seen in Chapter 4 and [8, 10, 17, 30].

Example 5.5:
Let $f(t) = \sqrt{2r}t$ and $L(t) \equiv L > 0$. Then we have extinction almost surely — and the

same applies to any f such that $\lim_{t\to\infty} t^{-1}\int_0^t f'(s)^2 ds \to 2r$ when we take fixed L. We are able to give much more interesting results along the same lines in Section 5.8.

Example 5.6:
Let $f(t) = \lambda(t+1)\sin(\log(t+1))$ and $L(t) \equiv L$. If $r < \frac{\lambda^2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) + \frac{\pi^2}{8L^2}$ then we have extinction almost surely; if $r > \frac{\lambda^2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) + \frac{\pi^2}{8L^2}$ then, on survival, the number of particles alive at time t oscillates, with

$$\liminf_{t\to\infty} \frac{1}{t}\log|\hat{N}(t)| = r - \frac{\pi^2}{8L^2} - \frac{\lambda^2}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log|\hat{N}(t)| = r - \frac{\pi^2}{8L^2} - \frac{\lambda^2}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right).$$

(Note the appearance of the golden ratio.)

The reason for this oscillation on the exponential scale becomes clearer when we consider the following simpler, but perhaps less natural, example.

Example 5.7:
Define a continuous function $f : [0,\infty) \to \mathbb{R}$ by setting $f(t) = 0$ for $t \in [0,1]$ and

$$f'(t) = \begin{cases} 0 & \text{if } \ 2^{2k} \le t < 2^{2k+1} \ \text{ for some } \ k \in \{0,1,2,\ldots\} \\ 1 & \text{if } \ 2^{2k+1} \le t < 2^{2k+2} \ \text{ for some } \ k \in \{0,1,2,\ldots\} \end{cases}.$$

Then, provided that $r > \frac{1}{3} + \frac{\pi^2}{8L^2}$, on non-extinction we have

$$\liminf_{t\to\infty} \frac{1}{t}\log|\hat{N}(t)| = r - \frac{\pi^2}{8L^2} - \frac{1}{3}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log|\hat{N}(t)| = r - \frac{\pi^2}{8L^2} - \frac{1}{6}.$$

The idea here is that the number of particles grows quickly when $f'(t) = 0$, but much more slowly when $f'(t) = 1$ as the steep gradient means that particles have to struggle to follow the path for a long time. As the size of the intervals $[2^n, 2^{n+1}]$ grows exponentially, the behaviour of the number of particles at time t is dominated by the behaviour on the most recent such interval. [We note that this choice of f is not twice differentiable; however, it can be uniformly approximated by twice differentiable functions, and it is easily checked that our results still hold - see Section 5.7.]

5.4 The spine setup

We use the BBM formulation seen in Chapter 2, in the special case $A \equiv 1$. In the interests of keeping this chapter reasonably self-contained, we summarise the setup here. We consider a one-dimensional binary branching Brownian motion, branching at rate r, with associated probability measure $\mathbb{P}$ under which

- we begin with a root particle, $\emptyset$, at 0;
- if a particle u is in the tree then all its ancestors are also in the tree (if v is an ancestor of u then we write $v < u$);
- each particle u has a lifetime σ_u, which is exponentially distributed with parameter r, and a fission time $S_u = \sum_{v \leq u} \sigma_v$;
- each particle u has a position $X_u(t) \in \mathbb{R}$ at each time $t \in [S_u - \sigma_u, S_u)$;
- at the fission time S_u, u has disappeared and been replaced by two children $u0$ and $u1$, which inherit the position of their parent;
- given its birth time and position, each particle u, while alive, moves according to a standard Brownian motion started from $X_u(S_u - \sigma_u)$ independently of all other particles.

For convenience, we extend the position of a particle u to all times $t \in [0, S_u)$, to include the paths of all its ancestors:

$$X_u(t) := X_v(t) \text{ if } v \leq u \text{ and } S_v - \sigma_v \leq t < S_v.$$

We recall that we defined $N(t)$ to be the set of particles alive at time t,

$$N(t) := \{u : S_u - \sigma_u \leq t < S_u\},$$

and also that

$$\hat{N}(t) := \{u \in N(t) : |X_u(s) - f(s)| < L(s) \;\; \forall s \leq t\}.$$

We choose from our BBM one distinguished line of descent or *spine* – that is, a subset ξ of the tree such that $\xi \cap N(t)$ contains exactly one particle for each t and if $u \in \xi$ and $v < u$ then $v \in \xi$. We make this choice as follows:

- the initial particle $\emptyset$ is in the spine;
- at the fission time of node u in the spine, the new spine particle is chosen uniformly at random from the two children $u0$ and $u1$ of u.

We denote the position of the spine particle at time t by ξ_t; however we may also occasionally use ξ_t to refer to the spine particle itself (that is, the node of the tree that is in the spine at time t) — it should be clear from the context which meaning is intended. We call the resulting probability measure (on the space of *marked trees with spines*) $\tilde{\mathbb{P}}$. We also consider the translated probability measures $\mathbb{P}_x$ and $\tilde{\mathbb{P}}_x$ for $x \in \mathbb{R}$, where under $\mathbb{P}_x$ and $\tilde{\mathbb{P}}_x$ we start with a single particle at x instead of 0.

5.4.1 Filtrations

We use three different filtrations, $\mathcal{F}_t$, $\tilde{\mathcal{F}}_t$ and $\mathcal{G}_t$, to encapsulate different amounts of information. We give descriptions of these filtrations here, but the reader is referred to Chapter 2 for the full definitions.

- $\mathcal{F}_t$ contains all the information about the marked tree up to time t. However, it does not know which particle is the spine at any point.
- $\tilde{\mathcal{F}}_t$ contains all the information about both the marked tree and the spine up to time t.
- $\mathcal{G}_t$ contains just the spatial information about the spine up to time t; it does not know anything about the rest of the tree.

We note that $\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t$ and $\mathcal{G}_t \subseteq \tilde{\mathcal{F}}_t$, and also that $\tilde{\mathbb{P}}_x$ is an extension of $\mathbb{P}_x$ in that $\mathbb{P}_x = \tilde{\mathbb{P}}_x|_{\mathcal{F}_\infty}$.

5.4.2 Martingales and a change of measure

Under $\tilde{\mathbb{P}}$, the path of the spine $(\xi_t,\ t \geq 0)$ is a standard Brownian motion. Set

$$\begin{aligned} G(t) := \exp\left(\int_0^t f'(s)d\xi_s - \frac{1}{2}\int_0^t f'(s)^2 ds + \int_0^t \frac{\pi^2}{8L(s)^2} ds\right) \\ \cdot \exp\left(\frac{L'(t)}{2L(t)}(\xi_t - f(t))^2 - \int_0^t \left(\frac{L''(s)}{2L(s)}(\xi_s - f(s))^2 + \frac{L'(s)}{2L(s)}\right) ds\right). \end{aligned}$$

We claim that the process

$$V(t) := G(t)\cos\left(\frac{\pi}{2L(t)}(\xi_t - f(t))\right), \ \ t \geq 0$$

is a $\mathcal{G}_t$-local martingale.

Lemma 5.8:
Let
$$F(t) := \exp\left(\int_0^t \frac{\pi^2}{8L(s)^2}ds + \frac{L'(t)}{2L(t)}\xi_t^2 - \int_0^t \left(\frac{L''(s)}{2L(s)}\xi_s^2 + \frac{L'(s)}{2L(s)}\right) ds\right).$$
The process
$$U(t) := F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)$$
is a $\mathcal{G}_t$-local martingale.

Proof. By Itô's formula,
$$\begin{aligned}
dU(t) = &\frac{\pi^2}{8L(t)^2}F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)dt \\
&+ \left(\frac{L''(t)}{2L(t)} - \frac{L'(t)^2}{2L(t)^2}\right)\xi_t^2 F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)dt \\
&- \left(\frac{L''(t)}{2L(t)}\xi_t^2 + \frac{L'(t)}{2L(t)}\right)F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)dt \\
&+ \frac{\pi L'(t)}{2L(t)^2}\xi_t F(t)\sin\left(\frac{\pi\xi_t}{2L(t)}\right)dt \\
&+ \frac{L'(t)}{L(t)}\xi_t F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)d\xi_t \\
&+ \frac{\pi}{2L(t)}F(t)\sin\left(\frac{\pi\xi_t}{2L(t)}\right)d\xi_t \\
&+ \left(\frac{L'(t)}{2L(t)} + \frac{L'(t)^2}{2L(t)^2}\xi_t^2\right)F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)dt \\
&- \frac{\pi^2}{8L(t)^2}F(t)\cos\left(\frac{\pi\xi_t}{2L(t)}\right)dt \\
&- \frac{\pi L'(t)}{2L(t)^2}\xi_t F(t)\sin\left(\frac{\pi\xi_t}{2L(t)}\right)dt.
\end{aligned}$$
□

Lemma 5.9:
The process $V(t)$, $t \geq 0$ is a $\mathcal{G}_t$-local martingale.

Proof. Again applying Itô's formula does the trick - or one may simply apply Girsanov's theorem to the result of Lemma 5.8. □

By stopping the process $V(t)$ at the first exit time of the spine particle from the tube $\{(x,t) : |f(t) - x| < L(t)\}$, we obtain also that
$$\zeta(t) := V(t)\mathbb{1}_{\{|f(s)-\xi_s|<L(s)\ \forall s\leq t\}}$$
is a $\mathcal{G}_t$-local martingale, and in fact since its size is constrained it is easily seen to be

a $\mathcal{G}_t$-martingale by applying Lemma 3.6. We call this martingale ζ the *single-particle martingale*.

Definition 5.10:
We define an $\tilde{\mathcal{F}}_t$-adapted martingale by

$$\tilde{\zeta}(t) = 2^{n(\xi,t)} \times e^{-rt} \times \zeta(t),$$

where $n(\xi,t) := |\{v : v < \xi_t\}|$ is the generation of the spine at time t. The proof that this process is an $\tilde{\mathcal{F}}_t$-martingale can be found in Chapter 2.

We note that if f is an $\tilde{\mathcal{F}}_t$-measurable function then we can write:

$$f(t) = \sum_{u \in N_t} f_u(t) \mathbb{1}_{\xi_t = u} \tag{5.1}$$

where each f_u is $\mathcal{F}_t$-measurable – intuitively, if f is in fact $\mathcal{G}_t$-measurable, one replaces every appearance of ξ_t with $X_u(t)$: so for example

$$\begin{aligned} G_u(t) := \exp&\left(\int_0^t f'(s) dX_u(s) - \frac{1}{2}\int_0^t f'(s)^2 ds + \int_0^t \frac{\pi^2}{8L(s)^2} ds\right) \\ &\cdot \exp\left(\frac{L'(t)}{2L(t)}(X_u(t) - f(t))^2 - \int_0^t \left(\frac{L''(s)}{2L(s)}(X_u(s) - f(s))^2 + \frac{L'(s)}{2L(s)}\right) ds\right). \end{aligned}$$

It is also shown in Chapter 2 that if we define

$$Z(t) := \sum_{u \in N(t)} e^{-rt} \zeta_u(t),$$

where ζ_u is the $\mathcal{F}_t$-adapted process defined via the representation of ζ as in (5.1), then

$$Z(t) = \tilde{\mathbb{P}}[\tilde{\zeta}(t) | \mathcal{F}_t]$$

and hence that Z is an $\mathcal{F}_t$-martingale. This martingale is the main object of interest.

Definition 5.11:
We define a new measure, $\tilde{\mathbb{Q}}_x$, via

$$\left.\frac{d\tilde{\mathbb{Q}}_x}{d\tilde{\mathbb{P}}_x}\right|_{\tilde{\mathcal{F}}_t} = \frac{\tilde{\zeta}(t)}{\tilde{\zeta}(0)}.$$

Also, for convenience, define $\mathbb{Q}_x$ to be the projection of the measure $\tilde{\mathbb{Q}}$ onto $\mathcal{F}_\infty$; then

$$\left.\frac{d\mathbb{Q}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{Z(t)}{Z(0)}.$$

Lemma 5.12:
Under $\tilde{\mathbb{Q}}_x$,

- *when at position y at time t the spine ξ moves as a Brownian motion with drift*

$$f'(t) + (y - f(t))\frac{L'(t)}{L(t)} - \frac{\pi}{2L(t)}\tan\left(\frac{\pi}{2L(t)}(y - f(t))\right);$$

- *the fission times along the spine occur at an accelerated rate $2r$;*
- *at the fission time of node v on the spine, the single spine particle is replaced by two children, and the new spine particle is chosen uniformly from the two children;*
- *the remaining child gives rise to an independent subtree, which is not part of the spine and which (along with its descendants) draws out a marked tree determined by an independent copy of the original measure $\mathbb{P}$ shifted to its position and time of birth.*

This, again, was covered in Chapter 2. We also use that, under $\tilde{\mathbb{Q}}_x$, the spine remains within distance $L(t)$ of $f(t)$ for all times $t \geq 0$. Intuitively, the tangent term gives an infinite drift away from the edges of the tube; but to see the proof explicitly, note that

$$\tilde{\mathbb{Q}}_x(\xi_t \notin \hat{N}(t)) = \tilde{\mathbb{P}}_x\left[\mathbb{1}_{\{\xi_t \notin \hat{N}(t)\}}\frac{\tilde{\zeta}(t)}{\tilde{\zeta}(0)}\right] = 0$$

by definition of $\tilde{\zeta}(t)$. All other particles, once born, move like independent standard Brownian motions but – as under $\mathbb{P}_x$ – we imagine them being "killed" instantly upon leaving the tube of radius L about f. In reality they are still present in the system, but make no contribution to Z once they have left the tube.

Remark:
Note that $\hat{N}$, and hence Z, $\tilde{\mathbb{Q}}$ and various other of our constructions, depend upon the choice of function f and radius L. Usually these will be implicit, but occasionally we shall write $\hat{N}^{f,L}$, $Z^{f,L}$ and $\tilde{\mathbb{Q}}^{f,L}$ (and so on) to emphasise the choice of f and L in use at the time.

We will, as usual, find the spine decomposition theorem to be a vital tool in our investigation. However for this chapter we will need only the following simplified form.

Theorem 5.13 (Spine decomposition)**:**
We have the following decomposition of Z:

$$\tilde{\mathbb{Q}}_x[Z(t)|\mathcal{G}_\infty] = \int_0^t 2re^{-rs}\zeta(s)ds + e^{-rt}\zeta(t).$$

Proof. We know from Theorem 2.7 that

$$\tilde{\mathbb{Q}}[Z(t)|\tilde{\mathcal{G}}_\infty] = \sum_{u<\xi_t} e^{-rS_u}\zeta(S_u) + e^{-rt}\zeta(t).$$

Conditioning now on $\mathcal{G}_\infty$, under $\tilde{\mathbb{Q}}$ the births along the spine form a Poisson process of rate $2r$ and hence the sum collapses to an integral (see for example [24]) to give the result. □

5.5 Almost sure growth along paths

5.5.1 Controlling the measure change

Before applying the tools that we have developed, we need the following short lemma to keep the Girsanov part of our change of measure under control.

Lemma 5.14:
For any $u \in \hat{N}(t)$, almost surely under both $\tilde{\mathbb{P}}_x$ and $\tilde{\mathbb{Q}}_x$ we have

$$\left|\int_0^t f'(s)dX_u(s) - \int_0^t f'(s)^2 ds\right| \leq |f'(t)|L(t) + |f'(0)|x + \int_0^t |f''(s)|L(s)ds$$

and hence under $\tilde{\mathbb{P}}$

$$\exp\left(\frac{1}{2}\int_0^t f'(s)^2ds + \int_0^t \frac{\pi^2}{8L(s)^2}ds - \int_0^t \frac{L'(s)}{2L(s)}ds - E(t)\right)$$
$$\leq G_u(t) \leq \exp\left(\frac{1}{2}\int_0^t f'(s)^2ds + \int_0^t \frac{\pi^2}{8L(s)^2}ds - \int_0^t \frac{L'(s)}{2L(s)}ds + E(t)\right). \quad (5.2)$$

Proof. From the integration by parts formula for Itô calculus, we know that

$$f'(t)X_u(t) = f'(0)X_u(0) + \int_0^t f''(s)X_u(s)ds + \int_0^t f'(s)dX_u(s).$$

From ordinary integration by parts,

$$\int_0^t f'(s)^2ds = f'(t)f(t) - f'(0)f(0) - \int_0^t f(s)f''(s)ds.$$

We also note that if $u \in \hat{N}(t)$ then $|X_u(s) - f(s)| < L(s)$ for all $s \leq t$. Thus

$$\begin{aligned}
&\left| \int_0^t f'(s) dX_u(s) - \int_0^t f'(s)^2 ds \right| \\
&= \left| f'(t)(X_u(t) - f(t)) - f'(0)(X_u(0) - f(0)) - \int_0^t f''(s)(X_u(s) - f(s)) ds \right| \\
&\leq |f'(t)| L(t) + |f'(0)| x + \int_0^t |f''(s)| L(s) ds.
\end{aligned}$$

Plugging this estimate into the definition of $G_u(t)$ gives the result. □

We are now ready to prove our first real result.

Proposition 5.15:
Recall that $Z(\infty) := \limsup_{t\to\infty} Z(t)$. If $S < 0$, then the process almost surely becomes extinct in finite time (and hence we have $Z(\infty) = 0$). In this case,

$$\frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{\inf_{s\leq t} \int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2} f'(u)^2 + \frac{L'(u)}{2L(u)}) du} \longrightarrow 1.$$

Alternatively, if $S > 0$ then $\mathbb{P}[Z(\infty)] = 1$.

Proof. We first recall the spine decomposition and apply inequality (5.2):

$$\begin{aligned}
\tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty] &= \int_0^t 2r e^{-rs} \zeta(s) ds + e^{-rt} \zeta(t) \\
&\leq \int_0^t 2r e^{-\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2} f'(u)^2 + \frac{L'(u)}{2L(u)}) du + E(s)} ds \\
&\qquad + e^{-\int_0^t (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2} f'(u)^2 + \frac{L'(u)}{2L(u)}) du + E(t)}.
\end{aligned}$$

If $S > 0$, then the integrand above is exponentially small for all large t (as is the second term); so $\liminf_{t\to\infty} \tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty] < \infty$. By Proposition 3.2 we know that $1/Z$ is a positive $(\tilde{\mathbb{Q}}, \mathcal{F}_t)$-supermartingale, and hence $Z(t)$ converges $\tilde{\mathbb{Q}}$-almost surely to some (possibly infinite) limit. Thus, applying Fatou's lemma, we get

$$\tilde{\mathbb{Q}}[Z(\infty)|\mathcal{G}_\infty] \leq \liminf_{t\to\infty} \tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty] < \infty.$$

We deduce that $Z(\infty) < \infty$ $\tilde{\mathbb{Q}}$-almost surely, and Lemma 2.8 then gives that $\mathbb{P}[Z(\infty)] = 1$.

Alternatively, suppose that $S < 0$. Then by the above,

$$\tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty] \leq (2rt + 1) e^{-\inf_{s\leq t} \left\{ \int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2} f'(u)^2 + \frac{L'(u)}{2L(u)}) du - E(s) \right\}}.$$

Now, by the tower property of conditional expectation and Jensen's inequality,

$$\mathbb{P}(\hat{N}(t) \neq \emptyset) = \mathbb{P}(Z(t) > 0) = \mathbb{Q}\left[\frac{1}{Z(t)}\right] \geq \tilde{\mathbb{Q}}\left[\frac{1}{\tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty]}\right].$$

This clearly implies that, for large t (using that $S < 0$),

$$\frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{\inf_{s\leq t}\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du}$$
$$\leq \frac{\inf_{s\leq t}\left\{\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du - E(s)\right\} - \log(2rt+1)}{\inf_{s\leq t}\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du};$$

and it is easy to see that the right-hand side converges to one as $t \to \infty$. This gives us our upper bound.

For the lower bound (still in the case $S < 0$), suppose for a moment that we may choose $\gamma > 1$ such that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8\gamma L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds < 0.$$

We note that we may choose γ in this way if $\int_0^t \pi^2/8L(s)^2 ds$ (eventually) shows at most linear growth, which we will check later. Then

$$\begin{aligned}\mathbb{P}(\hat{N}(t) \neq \emptyset) = \inf_{s\leq t}\mathbb{P}(\hat{N}(s) \neq \emptyset) &= \inf_{s\leq t}\mathbb{P}\left[\frac{Z^{f,\gamma L}(s)}{Z^{f,\gamma L}(s)}\mathbb{1}_{\{\hat{N}^{f,L}(s)\neq\emptyset\}}\right]\\ &= \inf_{s\leq t}\mathbb{Q}^{f,\gamma L}\left[\frac{1}{Z^{f,\gamma L}(s)}\mathbb{1}_{\{\hat{N}^{f,L}(s)\neq\emptyset\}}\right]\\ &\leq \inf_{s\leq t}\mathbb{Q}^{f,\gamma L}\left[\frac{\mathbb{1}_{\{\hat{N}^{f,L}(s)\neq\emptyset\}}}{\sum_{v\in\hat{N}^{f,L}(s)} e^{-rs}\zeta_v^{f,\gamma L}(s)}\right].\end{aligned}$$

If $\hat{N}^{f,L}(s) \neq \emptyset$ then there is at least one particle v in $\hat{N}^{f,L}(s)$; we may then apply inequality (5.2) to $\zeta_v^{f,\gamma L}(s)$ see that

$$\mathbb{P}(\hat{N}(t) \neq \emptyset) \leq \inf_{s\leq t}\frac{1}{e^{-\int_0^s (r - \frac{\pi^2}{8\gamma^2 L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du - \gamma^2 E(s)}\cos(\pi/2\gamma)}.$$

We repeat our calculations from the upper bound, taking logarithms and dividing by the desired denominator, to give

$$
\begin{aligned}
&\frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{\inf_{s\leq t}\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du} \\
&\geq \frac{\inf_{s\leq t}\left\{\int_0^s (r - \frac{\pi^2}{8\gamma^2 L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du - \gamma^2 E(s)\right\} - \log\cos(\pi/2\gamma)}{\inf_{s\leq t}\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du} \\
&\geq 1 + \frac{\left(1-\frac{1}{\gamma^2}\right)\sup_{s\leq t}\int_0^s \frac{\pi^2}{8L(u)^2}du + \gamma^2 \sup_{s\leq t} E(s) - \log\cos(\pi/2\gamma)}{\inf_{s\leq t}\int_0^s (r - \frac{\pi^2}{8L(u)^2} - \frac{1}{2}f'(u)^2 + \frac{L'(u)}{2L(u)})du}
\end{aligned}
\tag{5.3}
$$

for large t. Thus it remains to check that the right-hand side above has a limsup that is close to 1 when γ is close to 1. Again it is sufficient that $\int_0^t \pi^2/8L(s)^2 ds$ can (eventually) show at most linear growth, and we check that fact now. This is rather fiddly and not interesting in the context of the rest of the proof. Suppose it is not true; that is, suppose

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \frac{\pi^2}{8L(s)^2} ds = \infty.$$

Then since $S > -\infty$ we must have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(\frac{\pi^2}{8L(s)^2} - \frac{L'(s)}{2L(s)}\right) ds < \infty. \tag{5.4}$$

If we take $T_n := \inf\{t > 0 : \int_0^t \pi^2/8L(s)^2 ds > nt\}$, then

$$\frac{d}{dt}\left(\frac{1}{t}\int_0^t \frac{\pi^2}{8L(s)^2}ds\right)\bigg|_{T_n} > 0,$$

so differentiating and rearranging we get

$$L(T_n)^2 < \frac{\pi^2 T_n}{8\int_0^{T_n} \frac{\pi^2}{8L(s)^2}ds} < \frac{\pi^2}{8n}.$$

Now, we note that $\int_0^t \frac{L'(s)}{L(s)} ds = \log L(t) - \log L(0)$, so (5.4) implies that for all large t,

$$\int_0^t \frac{\pi^2}{8L(s)^2} ds < Kt + \frac{1}{2}\log L(t)$$

for some constant K. We have just shown that $L(T_n)^2 < \pi^2/8n$, so for all large n,

$$\int_0^{T_n} \frac{\pi^2}{8L(s)^2} ds < KT_n + \frac{1}{4}\log\frac{\pi^2}{8n}$$

contradicting (for large n) the definition of T_n.

We have shown that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \frac{\pi^2}{8L(s)^2} ds < \infty;$$

which allows us to make the limsup of (5.3) as close to 1 as we like by letting $\gamma \downarrow 1$. This completes the lower bound, which in particular implies (by monotonicity) that the probability of eventual extinction is equal to 1. □

5.5.2 Almost sure growth

Having established, in Proposition 5.15, the large deviations behaviour of our model, we now turn to the question of what happens when extinction does not occur. The two propositions in this section contain the meat of our results in this direction. Proposition 5.16 gives a lower bound on the number of particles in $\hat{N}(t)$ for large t, and Proposition 5.17 an upper bound. The former holds only on the event that Z has a positive limit; as mentioned in the introduction, this set coincides (up to a null event) with the event that some particle manages to follow within L of f, although we will not prove this fact until Section 5.6. The proofs of our two propositions are very simple, but we stress again that this is due to the careful choice of martingale.

Proposition 5.16:
Let $\Omega^\star$ be the set on which Z has a strictly positive limit,

$$\Omega^\star := \left\{ \liminf_{t\to\infty} Z(t) > 0 \right\}.$$

If $S > 0$ then $\mathbb{P}$-almost surely on $\Omega^\star$ we have

$$\liminf_{t\to\infty} \frac{\log |\hat{N}(t)|}{\int_0^t \left(r - \frac{1}{2} f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)} \right) ds} \quad \geq \quad 1.$$

Proof. For any $t \geq 0$, by inequality (5.2), almost surely under $\mathbb{P}$

$$Z(t) = \sum_{u \in \hat{N}(t)} e^{-rt} \zeta_u(t) \leq |\hat{N}(t)| e^{-\int_0^t (r - \frac{\pi^2}{8L(s)^2} - \frac{1}{2} f'(s)^2 + \frac{L'(s)}{2L(s)}) ds + E(t)}.$$

Hence (for large t, since $S > 0$)

$$\frac{\log|\hat{N}(t)|}{\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds} \geq \frac{\log Z(t) + \int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds - E(t)}{\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds}.$$

Now, on $\Omega^\star$ we have $\liminf_{t\to\infty} Z(t) > 0$ and thus $\frac{1}{\delta t}\log Z(t)$ has a non-negative liminf for any $\delta > 0$; then since $S > 0$ we see that the right-hand side above has liminf at least 1. □

Remark:
Recall that under $\mathbb{P}$, Z is a non-negative martingale, and hence $\liminf_{t\to\infty} Z(t) = Z(\infty)$ $\mathbb{P}$-almost surely. If $S > 0$, then by Proposition 5.15 $\mathbb{P}[Z(\infty)] = 1$, so in this case $\Omega^\star$ occurs with strictly positive probability.

Proposition 5.17:
If $S > 0$, then $\mathbb{P}$-almost surely we have

$$\limsup_{t\to\infty} \frac{\log|\hat{N}(t)|}{\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds} \leq 1.$$

Proof. Fix $\gamma > 1$ and let $\varepsilon = \cos(\pi/2\gamma)$. Since $Z^{f,\gamma L}$ is a non-negative martingale under $\mathbb{P}$, we have $Z^{f,\gamma L}(\infty) < \infty$ $\mathbb{P}$-almost surely. This implies that for any $\delta > 0$, almost surely

$$\limsup_{t\to\infty} \frac{1}{\delta t}\log Z^{f,\gamma L}(t) \leq 0.$$

Now, almost surely under $\mathbb{P}$,

$$Z^{f,\gamma L}(t) = \sum_{u\in\hat{N}^{f,\gamma L}(t)} e^{-rt}\zeta_u^{f,\gamma L}(t) \geq \sum_{u\in\hat{N}^{f,L}(t)} e^{-rt}\zeta_u^{f,\gamma L}(t).$$

By the definition of ε above, for any $u \in \hat{N}^{f,L}(t)$ the cosine term in $\zeta_u^{f,\gamma L}(t)$ is at least ε (since the particle is within L of $f(t)$ at time t). Applying inequality (5.2) we see that

$$Z^{f,\gamma L}(t) \geq |\hat{N}^{f,L}| \cdot \varepsilon \cdot e^{-\int_0^t (r - \frac{\pi^2}{8\gamma^2 L(s)^2} - \frac{1}{2}f'(s)^2 + \frac{L'(s)}{2L(s)})ds - \gamma^2 E(t)}$$

and hence

$$\frac{\log |\hat{N}(t)|}{\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds} \\ \leq \frac{\log Z(t) - \log \varepsilon + \int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8\gamma^2 L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds + \gamma^2 E(t)}{\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds}.$$

As in Proposition 5.15, we can bound the growth of the $\int_0^t \frac{\pi^2}{8\gamma^2 L(s)^2} ds$ term in the numerator so that letting $\gamma \downarrow 1$ we get the desired result. $\square$

Corollary 5.18:
If $S > 0$, then $\mathbb{P}$-almost surely on the event $\Omega^\star$,

$$\frac{\log |\hat{N}(t)|}{\int_0^t (r - \frac{\pi^2 t}{8L^2} - \frac{1}{2}\int_0^t f'(s)^2 + \frac{L'(s)}{2L(s)}) ds} \longrightarrow 1.$$

Proof. Simply combine Propositions 5.16 and 5.17. $\square$

5.6 Showing that $Z(\infty) = 0$ agrees with extinction

We note that we have now established our main result except for one key point: our growth results have so far been on the event $\{Z(\infty) > 0\}$, rather than the event of survival of the process, $\{\Upsilon = \infty\}$. We turn now to showing that these two events differ only on a set of zero probability.

The approach to proving this is often analytic: one shows that $\mathbb{P}(Z(\infty) > 0)$ and $\mathbb{P}(\Upsilon = \infty)$ satisfy the same differential equation with the same boundary conditions, and then shows that any such solution to the equation is unique. There is also sometimes a probabilistic approach to such arguments: one considers the product martingale

$$P(t) := \mathbb{P}(Z(\infty) = 0 | \mathcal{F}_t) = \prod_{u \in N(t)} \mathbb{P}_{X_u(t)}(Z(\infty) = 0).$$

On extinction, the limit of this process is clearly 1, and if we could show that on survival the limit is 0, then since P is a bounded non-negative martingale we would have

$$\mathbb{P}(\Upsilon < \infty) = \mathbb{P}[P(\infty)] = \mathbb{P}[P(0)] = \mathbb{P}(Z(\infty) = 0).$$

In Harris *et al.* [15], for example, we have killing of particles at the origin rather than on the boundary of a tube – and it is shown that on survival, at least one particle escapes to infinity and its term in the product martingale tends to zero. This is enough to complete the argument (although in [15] the authors favour the analytic approach). In our case we are hampered by the fact that for a single particle u the value of $\mathbb{P}_{X_u(t)}(Z_u(\infty) = 0)$ is bounded away from zero, and if the particle is close to the edge of the tube, or even possibly in some places in the interior the tube, then this probability takes values arbitrarily close to 1.

The time-inhomogeneity of our problem means that other standard methods also fail. Our alternative approach is based upon similar principles as the probabilistic approach above, but is more direct: we show that if at least one particle survives for a long time, then it will have many births in "good" areas of the tube, and thus $Z(\infty) > 0$ with high probability.

Recall that under $\tilde{\mathbb{P}}_x$, we start at time $t = 0$ with one particle at position x (rather than at the origin) – and similarly for $\tilde{\mathbb{Q}}_x$. We assume throughout this section that $S > 0$, otherwise there is nothing to prove (our theorem does not consider the case $S = 0$, and if $S < 0$ we have proved that $\mathbb{P}(\Upsilon = \infty) = 0 = \mathbb{P}(Z(\infty) > 0)$). We now need some more notation.

Definition 5.19:
Let $L_0 := \frac{\pi}{2\sqrt{S}} \vee 1$, and define

$$\begin{array}{rccl} \tilde{L}: & [0,\infty) & \to & (0,\infty) \\ & t & \mapsto & \begin{cases} L(t) & \text{if } L(t) \le L_0 \\ L_0 + (L(t) - L_0)e^{-(L(t)-L_0)^2} & \text{if } L(t) > L_0 \end{cases} \end{array}$$

and

$$\begin{array}{rccl} \tilde{f}: & [0,\infty) & \to & \mathbb{R} \\ & t & \mapsto & f(t) + L(t) - \tilde{L}(t). \end{array}$$

Now, for any function g on $[0,\infty)$, define the t-delayed version g_t of g for $t \in [0,\infty)$ by

$$g_t(s) = g(t+s) - g(t), \quad s \ge 0.$$

Thus for each $t \ge 0$ we have four new functions f_t, $\tilde{f}_t$, L_t and $\tilde{L}_t$.

Also, for $\alpha \in [0,1)$, define

$$U_\alpha = \{(t,x) : \mathbb{P}_{x-f(t)}(Z^{f_t,L_t}(\infty) > 0) \ge \alpha\} \subseteq [0,\infty) \times \mathbb{R}.$$

We think of U_α as the "good" part of the tube — if a particle is born in U_α then it has probability at least α of contributing to $Z(\infty)$. Finally, for any particle u and $t \ge 0$, define

$$I_\alpha(u;t) = \int_0^{t\wedge S_u} \mathbb{1}_{\{X_u(s)\in U_\alpha\}} ds;$$

$I_\alpha(u;t)$ is the time spent by particle u in the set U_α before t.

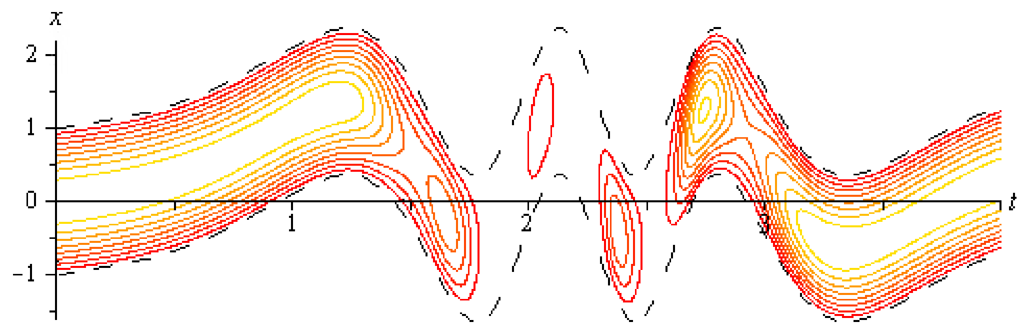

Figure 5-1: Approximation to a section of U_α for eight different values of α when $f(t) = \sin(a \tanh(t + b)) + c$ for some constants a, b and c.

Our first task is to convert to using $\tilde{f}$ and $\tilde{L}$; the fact that $\tilde{L}$ is bounded will prove useful.

Lemma 5.20:
The pair $(\tilde{f}, \tilde{L})$ satisfies usual conditions (II, III, IV), and $\tilde{S} := S^{\tilde{f},\tilde{L}} \geq S^{f,L}/2 > 0$.

Proof. We note that $\tilde{L}$ is twice continuously differentiable and hence so is $\tilde{f}$, and that $\tilde{L}(t) = L(t)$ whenever $L(t) \leq L_0$, $\tilde{L}(t) \geq L_0$ whenever $L(t) \geq L_0$, and $\tilde{L}(t) \leq L(t)\wedge(L_0+1)$ for all $t \geq 0$. We first claim that $E^{\tilde{f},\tilde{L}}(t) = o(t)$, working by comparison with $E^{f,L}$. Indeed, when $L(t) \leq L_0$ we clearly have $|\tilde{L}'(t)| = |L'(t)|$ and $|\tilde{L}''(t)| = |L''(t)|$. When $L(t) > L_0$,

$$\tilde{L}'(t) = L'(t)(1 - 2(L(t) - L_0)^2)e^{-(L(t)-L_0)^2}$$

so $|\tilde{L}'(t)| \leq |L'(t)|$. Also,

$$\begin{aligned}\tilde{L}''(t) = L''(t)e^{-(L(t)-L_0)^2} &- 6L'(t)^2(L(t) - L_0)e^{-(L(t)-L_0)^2} \\ &- 2L''(t)(L(t) - L_0)^2e^{-(L(t)-L_0)^2} + 4L'(t)^2(L(t) - L_0)^3e^{-(L(t)-L_0)^2}\end{aligned}$$

so (since for $x \geq 0$ the sizes of xe^{-x^2}, $x^2e^{-x^2}$ and $x^3e^{-x^2}$ are bounded above by 1)

$$\begin{aligned}\int_0^t |\tilde{L}''(s)|\tilde{L}(s)ds \leq \int_0^t |L''(s)|L(s)ds &+ 6(L_0 + 1)\int_0^t L'(s)^2ds \\ &+ 2\int_0^t |L''(s)|L(s)ds + 4(L_0 + 1)\int_0^t L'(s)^2ds.\end{aligned}$$

Each of these terms on the right-hand side above is $o(t)$ since

$$\int_0^t L'(s)^2 ds = L'(t)L(t) - L'(0)L(0) - \int_0^t L''(s)L(s)ds$$

and L satisfies our usual conditions. As $\tilde{f}'(t) = f'(t) + L'(t) - \tilde{L}'(t)$, and similarly for $\tilde{f}''$, we may also bound $|\tilde{f}'(t)|\tilde{L}(t)$ and $\int_0^t |\tilde{f}''(s)|\tilde{L}(s)ds$ simply by using the above estimates along with the triangle inequality and linearity of the integral. Thus, provided that $E^{f,L}(t) = o(t)$ we must have $E^{\tilde{f},\tilde{L}}(t) = o(t)$. Clearly also $S^{\tilde{f},\tilde{L}} \in (-\infty, \infty)$.

Secondly, we claim that $\limsup_{t\to\infty} \frac{1}{t} \log L(t) \leq 0$. Suppose not; then there exist $\varepsilon > 0$ and $t_n \to \infty$ such that $L(t_n) > e^{\varepsilon t_n}$ for each n. Setting

$$T_n := \sup\{t \in [0, t_n) : L(t) < e^{\varepsilon t_n}/2\},$$

if $T_n > 0$ (which must occur for all but finitely many n) then by the mean value theorem we can choose $c_n \in (T_n, t_n)$ such that $L'(c_n) \geq e^{\varepsilon t_n}/2t_n$. But $L(c_n) \geq e^{\varepsilon t_n}/2$, so $L'(c_n)L(c_n) \geq e^{2\varepsilon t_n}/4t_n$, contradicting the assumption that (f, L) satisfies the usual conditions (specifically the requirement that $L'(t)L(t) = o(t)$).

Thirdly, we show that $\int_0^t \tilde{f}'(s)^2 ds = \int_0^t f'(s)^2 ds + o(t)$. By Minkowski's inequality,

$$\begin{aligned}\left(\int_0^t \tilde{f}'(s)^2 ds\right)^{1/2} &= \left(\int_0^t (f'(s) + L'(s) - \tilde{L}'(s))^2 ds\right)^{1/2} \\ &\leq \left(\int_0^t f'(s)^2 ds\right)^{1/2} + \left(\int_0^t L'(s)^2 ds\right)^{1/2} + \left(\int_0^t \tilde{L}'(s)^2 ds\right)^{1/2}\end{aligned}$$

but

$$\int_0^t L'(s)^2 ds = L(t)L'(t) - L(0)L'(0) - \int_0^t L''(s)L(s)ds = o(t)$$

and the same calculation holds for $\tilde{L}$. Similarly by writing out $(\int_0^t f'(s)^2 ds)^{1/2}$ in terms of $\tilde{f}'$, L' and $\tilde{L}$ and applying Minkowski's inequality we get that

$$\int_0^t f'(s)^2 ds \leq \int_0^t \tilde{f}'(s)^2 ds + o(t).$$

Our final claim is that $\tilde{S} := S^{\tilde{f},\tilde{L}} \geq S^{f,L}/2 > 0$. Indeed, using various facts just established,

$$
\begin{aligned}
&\frac{1}{t}\int_0^t \left(r - \frac{1}{2}\tilde{f}'(s)^2 - \frac{\pi^2}{8\tilde{L}(s)^2} + \frac{\tilde{L}'(s)}{\tilde{L}(s)} \right) ds \\
&\geq \frac{1}{t}\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} \right) ds - \frac{1}{t}\int_0^t \frac{\pi^2}{8L_0^2} ds + \frac{1}{t}\log \tilde{L}(t) - \frac{1}{t}\log \tilde{L}(0) + o(1) \\
&\geq \frac{1}{t}\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} \right) ds - S/2 + \frac{1}{t}\log(L(t) \wedge 1) - \frac{1}{t}\log \tilde{L}(0) + o(1)
\end{aligned}
$$

so that (since $\limsup \frac{1}{t}\log L(t) \leq 0$)

$$
\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(r - \frac{1}{2}\tilde{f}'(s)^2 - \frac{\pi^2}{8\tilde{L}(s)^2} + \frac{\tilde{L}'(s)}{\tilde{L}(s)} \right) ds \\
&\geq \liminf_{t\to\infty} \left\{ \frac{1}{t}\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8\tilde{L}(s)^2} \right) ds + \frac{1}{t}\log L(t) \right\} - S/2 \\
&\geq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{L(s)} \right) ds - S/2 \\
&= S^{f,L}/2
\end{aligned}
$$

as required. □

Our next lemma establishes that for sufficiently small α, U_α — which we think of as the good part of the tube — stretches to near the top and bottom edges of the L-tube for almost $S/2r$ proportion of the time. To do this we use the identity given in Lemma 3.4 combined with the spine decomposition. For $\delta \in (0,1)$ and $t \geq 0$, let

$$\hat{L}(t) := ((1-\delta)L(t)) \vee (L(t) - \delta).$$

Lemma 5.21:
Fix $\delta \in (0,1)$ and $\beta < 1$. If $S > 0$ then for sufficiently small $\alpha > 0$ and large T, we have

$$\int_0^t \mathbb{1}_{\{(s,x)\in U_\alpha \ \forall x \in [f(s)-\hat{L}(s), f(s)+\hat{L}(s)]\}} ds \geq \beta \frac{S}{2r} t \ \ \forall t \geq T.$$

Proof. Fix $q \in (0, \frac{1-\beta}{3})$ and $p \in (\beta + 3q, 1)$; we show that for

$$\alpha = \frac{q\tilde{S}\cos(\pi\delta/2)}{2re^{(L_0+1)(r\sqrt{2/q\tilde{S}}+1)}}$$

and all sufficiently large t we have

$$\int_0^t \mathbb{1}_{\{(s,x)\in U_\alpha\ \forall x\in[f(t)-\hat{L}(s), f(t)+\hat{L}(s)]\}} ds \geq (p-3q)\frac{S}{2r}t.$$

We begin working with $\tilde{f}$ and $\tilde{L}$; we shall move back to f and L towards the end of the proof. Let

$$J_t = \inf_{s\geq t}\left\{\int_0^s \left(r - \frac{\pi^2}{8\tilde{L}(u)^2} - \frac{1}{2}\tilde{f}'(u)^2 + \frac{\tilde{L}'(u)}{2\tilde{L}(u)} - q\tilde{S}\right) du - E^{\tilde{f},\tilde{L}}(s)\right\},$$

and define three subsets, U, V and W, of $[0,\infty)$ by

$$U = \{t \geq 0 : J_t \text{ is increasing at } t\},\ \ V = \left\{t\geq 0 : |\tilde{f}'(t)| < r\sqrt{2/q\tilde{S}}\right\}$$

and

$$W = \{t \geq 0 : |\tilde{L}'(t)| \leq 1\}.$$

If J is increasing at t, then clearly for any $s > 0$

$$\int_0^{t+s} \left(r - \frac{\pi^2}{8\tilde{L}(u)^2} - \frac{1}{2}\tilde{f}'(u)^2 + \frac{\tilde{L}'(u)}{2\tilde{L}(u)} - q\tilde{S}\right) du - E^{\tilde{f},\tilde{L}}(t+s)$$
$$> \int_0^{t} \left(r - \frac{\pi^2}{8\tilde{L}(u)^2} - \frac{1}{2}\tilde{f}'(u)^2 + \frac{\tilde{L}'(u)}{2\tilde{L}(u)} - q\tilde{S}\right) du - E^{\tilde{f},\tilde{L}}(t),$$

and hence

$$\int_t^{t+s} \left(r - \frac{\pi^2}{8\tilde{L}(u)^2} - \frac{1}{2}\tilde{f}'(u)^2 + \frac{\tilde{L}'(u)}{2\tilde{L}(u)}\right) du - E^{\tilde{f},\tilde{L}}(t+s) + E^{\tilde{f},\tilde{L}}(t) > q\tilde{S}s.$$

Thus if $t \in U\cap V\cap W$ then, as in Proposition 5.15, we can apply the spine decomposition, the fact that ζ is a non-negative martingale and thus has a finite limit almost surely, and

Lemma 5.14 to get, for any $x \in (-\tilde{L}(t), \tilde{L}(t))$,

$$\begin{aligned}
\tilde{\mathbb{Q}}_x^{\tilde{f}_t,\tilde{L}_t}[Z^{\tilde{f}_t,\tilde{L}_t}(\infty)|\mathcal{G}_\infty] &= \int_0^\infty 2re^{-rs}\zeta^{\tilde{f}_t,\tilde{L}_t}(s)ds + \lim_{t\to\infty} e^{-rt}\zeta^{\tilde{f}_t,\tilde{L}_t}(t) \\
&\le \int_0^\infty 2re^{-\int_0^s (r - \frac{\pi^2}{8\tilde{L}_t(u)^2} - \frac{1}{2}\tilde{f}_t'(u)^2 + \frac{\tilde{L}_t'(u)}{2\tilde{L}_t(u)})du + E^{\tilde{f}_t,\tilde{L}_t}(s)} ds \\
&\le \int_0^\infty 2re^{-\int_t^{t+s} (r - \frac{\pi^2}{8\tilde{L}(u)^2} - \frac{1}{2}\tilde{f}'(u)^2 + \frac{\tilde{L}'(u)}{2\tilde{L}(u)})du} \\
&\qquad \cdot e^{E^{\tilde{f},\tilde{L}}(t+s) - E^{\tilde{f},\tilde{L}}(t) + |\tilde{f}'(t)|\tilde{L}(t) + \frac{1}{2}|\tilde{L}'(t)|\tilde{L}(t)} ds \\
&\le e^{|\tilde{f}'(t)|\tilde{L}(t) + \frac{1}{2}|\tilde{L}'(t)|\tilde{L}(t)} \int_0^\infty 2re^{-q\tilde{S}s} ds \\
&\le \frac{2r}{q\tilde{S}} e^{(r\sqrt{2/q\tilde{S}}+1/2)(L_0+1)}
\end{aligned}$$

Using the identity from Lemma 3.4 together with Jensen's inequality gives that for any $x \in [\tilde{f}(t) - (((1-\delta)\tilde{L}(t)) \vee (\tilde{L}(t) - \delta)), \tilde{f}(t) + ((1-\delta)\tilde{L}(t)) \vee (\tilde{L}(t) - \delta)]$,

$$\begin{aligned}
\mathbb{P}_x(Z^{\tilde{f}_t,\tilde{L}_t}(\infty) > 0) &= \mathbb{Q}_x^{\tilde{f}_t,\tilde{L}_t}\left[\frac{Z^{\tilde{f}_t,\tilde{L}_t}(0)}{Z^{\tilde{f}_t,\tilde{L}_t}(\infty)}\right] \\
&\ge \tilde{\mathbb{Q}}_x^{\tilde{f}_t,\tilde{L}_t}\left[\tilde{\mathbb{Q}}_x^{\tilde{f}_t,\tilde{L}_t}\left[\frac{1}{Z^{\tilde{f}_t,\tilde{L}_t}(\infty)}\middle|\mathcal{G}_\infty\right]\right] e^{-\frac{1}{2}|\tilde{L}'(t)|\tilde{L}(t)} \cos\left(\frac{\pi x}{2\tilde{L}(t)}\right) \\
&\ge \tilde{\mathbb{Q}}_x^{\tilde{f}_t,\tilde{L}_t}\left[\frac{1}{\tilde{\mathbb{Q}}_x^{\tilde{f}_t,\tilde{L}_t}[Z^{\tilde{f}_t,\tilde{L}_t}(\infty)|\mathcal{G}_\infty]}\right] e^{-\frac{1}{2}L_0+1} \cos\left(\frac{\pi(L_0+1-\delta)}{2(L_0+1)}\right) \\
&\ge \frac{q\tilde{S}}{2re^{(r\sqrt{2/q\tilde{S}}+1)(L_0+1)}} \cos\left(\frac{\pi(L_0+1-\delta)}{2(L_0+1)}\right).
\end{aligned}$$

Now, since

$$\begin{aligned}
&[\tilde{f}(t) - (((1-\delta)\tilde{L}(t)) \vee (\tilde{L}(t) - \delta)), \tilde{f}(t) - (((1-\delta)\tilde{L}(t)) \vee (\tilde{L}(t) - \delta))] \\
&\qquad\qquad \supseteq [f(t) + L(t) - \tilde{L}(t) - \hat{L}(t), f(t) + \hat{L}(t)]
\end{aligned}$$

we have shown that if $t \in U \cap V \cap W$ then $\mathbb{P}_x(Z^{f_t,L_t}(\infty) > 0)$ is large enough for all $x \in [f(t) + L(t) - \tilde{L}(t) - \hat{L}(t), f(t) + \hat{L}(t)]$. If $x \in [f(t), f(t) + L(t) - \tilde{L}(t) - \hat{L}(t))$ then running the same argument as above but using $\tilde{f}^{(x)}(s) := \tilde{f}(s) - \tilde{f}(0) + x$, $s \ge 0$ in place of $\tilde{f}$ gives exactly the same result: so we have that $\mathbb{P}_x(Z^{f_t,L_t}(\infty) > 0)$ is large enough for the half-region $[f(t), f(t)+\hat{L}(t)]$ and by symmetry for the whole region $[f(t)-\hat{L}(t), f(t)+\hat{L}(t)]$.

Hence it now suffices to show that for large t,

$$\int_0^t \mathbb{1}_{U\cap V\cap W}(s)ds \geq (p-3q)\frac{S}{2r}t.$$

But for all large enough t, since J increases at rate at most r (recall that $\int_0^t \frac{\tilde{L}'(s)}{2\tilde{L}(s)}ds = \log\tilde{L}(t) - \log\tilde{L}(0)$, which is bounded) and $\lim_{t\to\infty} J_t = (1-q)\tilde{S}$,

$$(p-q)\tilde{S}t \leq J_t \leq \int_0^t r\mathbb{1}_U(s)ds.$$

Also, for large enough t we must have $\int_0^t \tilde{f}'(s)^2 ds \leq 2rt$ (otherwise $\tilde{S}$ would not be positive). Thus for large t

$$2rt \geq \int_0^t \tilde{f}'(s)^2 ds \geq \int_0^t \frac{2r^2}{q\tilde{S}}\mathbb{1}_{V^c}(s)ds;$$

finally,

$$\int_0^t \tilde{L}'(s)^2 ds = \tilde{L}(t)\tilde{L}'(t) - \tilde{L}(0)\tilde{L}'(0) + \int_0^t \tilde{L}(s)\tilde{L}''(s)ds$$

so since $E^{\tilde{f},\tilde{L}} = o(t)$ we have (again for large t)

$$\int_0^t \mathbb{1}_{W^c}(s)ds \leq \int_0^t \tilde{L}'(s)^2 ds \leq \frac{q\tilde{S}}{r}t.$$

Hence for all large t,

$$\begin{aligned}\int_0^t \mathbb{1}_{U\cap V\cap W}(s)ds &\geq \int_0^t \mathbb{1}_U(s)ds - \int_0^t \mathbb{1}_{V^c}(s)ds - \int_0^t \mathbb{1}_{W^c}(s)ds\\ &\geq (p-q)\frac{\tilde{S}}{r}t - q\frac{\tilde{S}}{r}t - q\frac{\tilde{S}}{r}t \geq (p-3q)\frac{S}{2r}t\end{aligned}$$

as required. $\square$

We now show that if a particle has remained in the tube for a long time, then it is very likely to have spent a long time in U_α. The idea is that if U_α stretches to within δ of the edge of the tube for a proportion of time, then in order to stay out of U_α a particle must spend a long time in a tube of radius δ. We use simple estimates for the time spent by Brownian motion in such a tube and apply these to our problem via the many-to-one theorem (Theorem 2.10).

Lemma 5.22:

Fix $\beta < 1$ and $\gamma > 0$. If $S > 0$ then for sufficiently small $\alpha > 0$ and large T, we have

$$\mathbb{P}(\exists u \in \hat{N}(t) : I_\alpha(u;t) < \beta \frac{S}{2r} t) \leq e^{-\gamma t} \quad \forall t \geq T.$$

Proof. For any $\delta \in (0,1)$, by Lemma 5.21 we may choose $\alpha > 0$ and T such that

$$\int_0^t \mathbb{1}_{\{(s,x)\in U_\alpha \ \forall x \in [f(t)-L+\delta, f(t)+L-\delta]\}} ds \geq \left(\frac{1+\beta}{2}\right) \frac{S}{2r} t \ \ \forall t \geq T.$$

Then if the spine particle is to have spent less than $\beta \frac{S}{2r} t$ time in U_α (yet remained within the tube of width L) then it must have spent at least $(\frac{1-\beta}{2}) \frac{S}{2r} t$ within δ of the edge of the tube (provided that t is large enough). That is, for $t \geq T$, if we let

$$V_s^1 := (f(s) - L(s), f(s) - L(s) + \delta) \cup (f(s) + L(s) - \delta, f(s) + L(s))$$

then

$$\tilde{\mathbb{P}}\left(\xi_t \in \hat{N}(t), I_\alpha(\xi_t;t) < \beta \frac{S}{2r} t\right) \leq \tilde{\mathbb{P}}\left(\xi_t \in \hat{N}(t), \int_0^t \mathbb{1}_{\{\xi_s \in V_s^1\}} ds > \left(\frac{1-\beta}{2}\right) \frac{S}{2r} t\right).$$

In fact, using the fact that if $\xi_t \in \hat{N}(t)$ then we may apply two simple Girsanov measure changes and our usual estimates on them. The first will give the spine drift f', and the second will give it an extra drift L'. Letting

$$V_s^2 := (-L(s), -L(s) + \delta) \cup (L(s) - \delta, L(s))$$

we have

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\xi_t \in \hat{N}(t), I_\alpha(\xi_t; t) < \beta\frac{S}{2r}t\right) \\
&\le \tilde{\mathbb{P}}\left[\frac{\mathbb{1}_{\{|\xi_s|<L(s)\ \forall s\in[0,t]\}}}{e^{\int_0^t f'(s)d\xi_s - \frac{1}{2}\int_0^t f'(s)^2 ds}}\mathbb{1}_{\{\int_0^t \mathbb{1}_{\{\xi_s\in V_s^2\}}ds > (\frac{1-\beta}{2})\frac{S}{2r}t\}}\right] \\
&\le e^{|f'(t)|L(t)+\int_0^t |f''(s)|L(s)ds} \\
&\qquad\qquad \cdot \tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s\in[0,t], \int_0^t \mathbb{1}_{\{\xi_s\in V_s^2\}}ds > \left(\frac{1-\beta}{2}\right)\frac{S}{2r}t\right) \\
&\le 2e^{|f'(t)|L(t)+\int_0^t |f''(s)|L(s)ds} \\
&\qquad\qquad \cdot \tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s\in[0,t], \int_0^t \mathbb{1}_{\{\xi_s\in(L(s)-\delta, L(s))\}}ds > \left(\frac{1-\beta}{2}\right)\frac{S}{4r}t\right) \\
&\le 2e^{|f'(t)|L(t)+\int_0^t |f''(s)|L(s)ds} \\
&\qquad\qquad \cdot \tilde{\mathbb{P}}\left[\frac{\mathbb{1}_{|\xi_s|<2L(s)\ \forall s\in[0,t]}}{e^{\int_0^t L'(s)d\xi_s - \frac{1}{2}\int_0^t L'(s)^2 ds}}\mathbb{1}_{\{\int_0^t \mathbb{1}_{\{\xi_s\in(-\delta,0)\}}ds > (\frac{1-\beta}{2})\frac{S}{4r}t\}}\right] \\
&\le 2e^{|f'(t)|L(t)+\int_0^t |f''(s)|L(s)ds + 2|L'(t)|L(t) + 2\int_0^t |L''(s)|L(s)ds} \\
&\qquad\qquad \cdot \tilde{\mathbb{P}}\left(\int_0^t \mathbb{1}_{\{\xi_s\in(-\delta,0)\}}ds > \left(\frac{1-\beta}{2}\right)\frac{S}{4r}t\right).
\end{aligned}$$

Using the estimate given in Lemma 4.11, and usual condition (III), we get that for large enough t

$$\tilde{\mathbb{P}}\left(\xi_t \in \hat{N}(t), I_\alpha(\xi_t; t) < \beta\frac{S}{2r}t\right) \le e^{(r+1)t - \frac{1}{4\delta}\left(\frac{1-\beta}{2}\right)\frac{S}{4r}t}.$$

Finally, taking $\delta = \frac{(1-\beta)S}{32r(2r+\gamma+1)}$ and using the many-to-one theorem (Theorem 2.10), for large t

$$\tilde{\mathbb{P}}\left(\exists u \in \hat{N}(t) : I_\alpha(u; t) < \beta\frac{S}{2r}t\right) \le e^{rt}\tilde{\mathbb{P}}\left(\xi_t \in \hat{N}(t), I_\alpha(\xi_t; t) < \beta\frac{S}{2r}t\right) \le e^{-\gamma t}. \qquad \square$$

We now combine the above results to achieve the aim of this section.

Proposition 5.23:
Recall that Υ is the extinction time for the process. If $S > 0$ then

$$\mathbb{P}(\Upsilon = \infty) = \mathbb{P}(Z(\infty) > 0).$$

Proof. We note that $\{Z(\infty) > 0\} \subseteq \{\Upsilon = \infty\}$, so it suffices to show that for any $\varepsilon > 0$,

$$\mathbb{P}(\Upsilon = \infty,\ \ Z(\infty) = 0) < \varepsilon.$$

To this end, fix $\varepsilon > 0$ and choose α small enough and T_0 large enough that

$$\mathbb{P}(\exists u \in \hat{N}(t) : I_\alpha(u;t) < \frac{S}{4r}t) < \varepsilon/3 \ \ \forall t \geq T_0$$

(this is possible by Lemma 5.22). Now choose an integer m large enough such that $(1-\alpha)^m < \varepsilon/3$. Finally, choose $T \geq T_0$ large enough that

$$\sum_{j=0}^{m-1} \frac{e^{-ST/4}(ST/4)^j}{j!} < \varepsilon/3.$$

Then

$$\begin{aligned}\mathbb{P}(\Upsilon = \infty, \ \ Z(\infty) = 0) &\leq \mathbb{P}(\exists u \in \hat{N}(T), \ \ Z(\infty) = 0) \\ &< \mathbb{P}\left(\exists u \in \hat{N}(T), \ I_\alpha(u;T) \geq \frac{S}{4r}T, \ Z(\infty) = 0\right) + \varepsilon/3.\end{aligned}$$

Now, if a particle u has spent at least $\frac{S}{4r}T$ time in U_α then (by the choice of T, since the births along u form a Poisson process of rate r) it has probability at least $(1-\varepsilon/3)$ of having at least m births whilst in U_α. Each of these particles born within U_α launches an independent population from a point $(t,x) \in U_\alpha$, so that

$$Z(\infty) \geq \sum_{v<u} e^{-r(S_v-\sigma_v)} Z_v(\infty) \mathbb{1}_{\{(S_v-\sigma_v, X_u(S_v-\sigma_v)) \in U_\alpha\}}$$

where each Z_v is a non-negative martingale on the interval $[S_v - \sigma_v, \infty)$ with law equal to that of Z^{f_t,L_t} started from x for some $(t,x) \in U_\alpha$, and hence satisfying $\mathbb{P}(Z_v(\infty) > 0) \geq \alpha$. Thus

$$\begin{aligned}&\mathbb{P}(\Upsilon = \infty, \ Z(\infty) = 0) \\ &\leq \mathbb{P}\left(\exists u \in \hat{N}(T), \ \ I_\alpha(u;T) \geq \frac{S}{4r}T, \ \ Z(\infty) = 0\right) + \varepsilon/3 \\ &\leq \mathbb{P}\left(\exists u \in \hat{N}(T), \ \left\{ \begin{array}{c} u \text{ has had at least} \\ m \text{ births within } U_\alpha \end{array} \right\}, \ \ Z(\infty) = 0\right) + 2\varepsilon/3 \\ &\leq (1-\alpha)^m + 2\varepsilon/3 \ \ < \ \ \varepsilon\end{aligned}$$

which completes the proof. □

We draw our results together as follows.

Proof of Theorem 5.1:
All that remains is to combine Proposition 5.15 with Corrolary 5.18 to gain the desired

growth bounds; Proposition 5.23 guarantees that we are working on the correct set. $\square$

5.7 Extending the class of functions

As promised, we can extend Theorem 5.1 to cover more general subsets of $C[0,\infty)$ in an obvious way: if a set $B \subset C[0,\infty)$ is contained within (or contains) an L-tube about a function f, then the set of particles with paths in B is a subset (respectively, superset) of the set of particles with paths within L of f, and if (f, L) satisfies our usual conditions then we have an immediate upper (lower) bound on the number of particles within B. That is, for any $B \subset C[0,\infty)$,

$$\sup \mathbb{P}(\hat{N}^{f,L}(t) \neq \emptyset) \leq \mathbb{P}(\hat{N}^{B}(t) \neq \emptyset) \leq \inf \mathbb{P}(\hat{N}^{f,L}(t) \neq \emptyset) \tag{5.5}$$

and

$$\sup |\hat{N}^{f,L}(t)| \leq |N^{B}(t)| \leq \inf |\hat{N}^{f,L}(t)| \tag{5.6}$$

where both suprema are taken over all f and L such that (f, L) satisfies our usual conditions and

$$\{g \in C[0,\infty) : |g(s) - f(s)| < L(s) \;\; \forall s \in [0,\infty)\} \subseteq B,$$

both infima are taken over all f and L such that (f, L) satisfies our usual conditions and

$$B \subseteq \{g \in C[0,\infty) : |g(s) - f(s)| < L(s) \;\; \forall s \in [0,\infty)\},$$

and

$$N^{B}(t) := \{u \in N(t) : \exists g \in B \text{ with } X_u(s) = g(s) \;\; \forall s \in [0,t]\}.$$

The obvious question now is whether this allows us to give growth rates for all sets in $C[0,\infty)$. The answer is no: there are still some seemingly reasonable sets that are not covered (which we shall see shortly).

Thus the natural question becomes whether we can instead characterise, in a more succinct way, the class of functions that Theorem 5.1 does cover, subject to using the extensions provided by (5.5) and (5.6). Can we weaken our usual conditions in some way that we can easily write down? The answer again seems to be, more or less, no. We may drop condition (I) as our eventual growth rate does not depend on the initial position of the particle as long as there is a path within our set that starts at the same point as the initial position of the first particle. We may also effectively drop condition (IV) — since it is not possible to get $S = \infty$ without violating condition (III), and the case $S = -\infty$ can always be covered either by bounding above using (5.5) and (5.6) or by using the many-to-one theorem, Theorem 2.10, more directly. However the interesting conditions (II) and (III) are difficult to shake off, a fact which is best demonstrated by a series of examples.

It is easiest to first consider condition (III).

Example 5.24:
Take $L(t) \equiv L > 0$ to be constant, and let

$$f_\delta(t) := \delta \sin(t/\delta);$$

then as $\delta \to 0$, f_δ converges uniformly to the zero function, $f(t) \equiv 0$. By Theorem 5.1 we know that on survival,

$$\lim_{t\to\infty} \frac{1}{t} \log |\hat{N}^{f,L}(t)| = r - \frac{\pi^2}{8L^2}.$$

However, if the result of Theorem 5.1 held for each f_δ then by approximation via (5.5) and (5.6) we would have (on survival)

$$\lim_{t\to\infty} \frac{1}{t} \log |\hat{N}^{f,L}(t)| = r - \frac{\pi^2}{8L^2} - \frac{1}{4}.$$

Of course, (f_δ, L) does not satisfy usual condition (III) and hence this contradiction does not appear – but the example shows that we cannot simply drop the requirement that $\int_0^t |f''(s)|L(s)ds = o(t)$.

Example 5.25:
Take $f(t) \equiv 0$ and $L(t) = 2 + \sin(t^{3/2})$. Intuitively, the sine term oscillates so fast for large t that we are effectively constrained within a tube of constant width 1. Thus we expect (and it is not too hard to imagine a hands-on proof using Theorem 5.1) that we should have a growth rate of $r - \pi^2/8$. However, one may show (for example by using the periodicity of sine and approximating the integral by a sum) that

$$\int_0^t \frac{1}{L(s)^2} ds \lesssim \frac{2t}{3\sqrt{3}}$$

so that if the result of Theorem 5.1 held in this case we would have a growth rate of at least $r - \pi^2/12\sqrt{3}$. Again, (f, L) does not satisfy usual condition (III) and we see that we cannot just drop the requirement that $\int_0^t |L''(s)|L(s)ds = o(t)$.

Example 5.26:
Take $f_0(t) \equiv 0$, $L_0(t) = \sqrt{t}$, $f_1(t) = t$ and $L_1(t) = t + \sqrt{t}$. Then the growth rate for (f_0, L_0) is r; and since the L_0-tube about f_0 is contained in the L_1-tube about f_1, we must have a growth rate for (f_1, L_1) of at least r (in fact it is exactly r since it is well-

known that the growth rate of the entire system is r). If the result of Theorem 5.1 held for (f_1, L_1) then its growth rate would be $r - 1/2$; so we see that we cannot simply drop the condition that $|f'(t)|L(t) + |L'(t)|L(t) = o(t)$.

Now consider condition (II). We can approximate any continuous function with twice continuously differentiable functions, but then how do we approach the conditions on the second derivative (from condition (III))? Even for constant L, there are some nowhere-differentiable paths f such that we may find a growth rate for $\hat{N}^{f,L}$ using (5.5) and (5.6), and some for which we may not. The lack of even a first derivative to work with in these cases precludes the existence of an obvious simple condition to tell us where to draw the line between these two groups. We claim simply that any non-smooth sets are best considered on a case-by-case basis using Theorem 5.1 together with (5.5) and (5.6).

For example, again with constant L, we may easily (by approximating by its partial sums) give a growth rate for the function

$$f(t) = \sum_{n=0}^{\infty} a^n (\cos(b^n \pi \log(t+1)) - 1)$$

(where b is a positive odd integer, $0 < a < 1$ and $ab > 1 + 3\pi/2$), which is a time change of a Weierstrass function and hence, by the chain rule, nowhere differentiable. On the other hand we cannot give an exact growth rate along (almost) any given Brownian path: any uniformly approximating functions must (by the fact that Brownian motion has independent increments) violate our conditions on the second derivative of f in (III).

5.8 The critical case $S = 0$

It would be remiss not to consider what can be done when $S = 0$. This is an interesting but delicate matter: our methods, as they stand, are not always sharp enough to say what will happen. There are several situations, however, where something can be done. Unfortunately we are again unable to provide a general theory, as our methods must be adapted carefully to the set in question. We give two such examples in Theorems 5.27 and 5.28 below.

Fix $\alpha > 0$, $\beta \in (0,1)$ and $\gamma > 0$, and for $t \geq 0$ let

$$f(t) = \alpha + \sqrt{2r}t - \alpha(t+1)^{\beta} \quad \text{and} \quad L(t) = \gamma(t+1)^{\beta}.$$

Theorem 5.27:

If $\beta < 1/3$ then we have $\mathbb{P}(\Upsilon = \infty) = 0$, and

$$\frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{t^{1-2\beta}} \longrightarrow -\frac{\pi^2}{8\gamma^2(1-2\beta)}.$$

If $\beta > 1/3$, we have $\mathbb{P}(\Upsilon = \infty) > 0$, and almost surely on survival

$$\liminf_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^\beta} \geq \alpha\sqrt{2r}$$

and

$$\limsup_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^\beta} \leq (\alpha+\gamma)\sqrt{2r}.$$

Proof. In the case $\beta < 1/3$ we may simply mimic the requisite part of the proof of Proposition 5.15, using the fact that for $\beta < 1/3$,

$$\int_0^t \left(r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)}\right) ds = \frac{\pi^2}{8\gamma^2(1-2\beta)}(t+1)^{1-2\beta} + o(t^{1-2\beta})$$

and

$$E(t) = \gamma\sqrt{2r}(t+1)^\beta + o(t^\beta).$$

Now suppose that $\beta > 1/3$. We proceed in very much the same way as in the main part of the article, leaving out many of the details. Direct calculation reveals that for $\beta > 1/3$,

$$\int_0^t (r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)})ds = \alpha\sqrt{2r}(t+1)^\beta + o(t^\beta)$$

and

$$E(t) = \gamma\sqrt{2r}(t+1)^\beta + o(t^\beta).$$

Thus, by the spine decomposition,

$$\tilde{\mathbb{Q}}[Z(t)|\mathcal{G}_\infty] \leq \int_0^t 2re^{-(\alpha-\gamma)\sqrt{2r}(s+1)^\beta + o(s^\beta)}ds + e^{-(\alpha-\gamma)\sqrt{2r}(t+1)^\beta + o(t^\beta)}$$

which converges as $t \to \infty$ provided that $\alpha > \gamma$. We deduce that $\mathbb{P}(Z(\infty) > 0) > 0$ provided that $\alpha > \gamma$, and indeed for all α and γ since for fixed α, increasing γ can only increase the probability of survival. The same argument as in Proposition 5.16 gives that on $\{Z(\infty) > 0\}$ we have

$$\liminf_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^\beta} \geq (\alpha-\gamma)\sqrt{2r},$$

and again we decide that since increasing γ can only increase $|\hat{N}(t)|$, we can take γ arbitrarily small and deduce that for any γ, on $\{Z(\infty) > 0\}$,

$$\liminf_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^\beta} \geq \alpha\sqrt{2r}.$$

The same argument as Proposition 5.17 also gives

$$\limsup_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^\beta} \leq (\alpha + \gamma)\sqrt{2r}.$$

We must now check that $\{Z(\infty) > 0\}$ agrees with $\{\Upsilon = \infty\}$ up to a set of zero probability. The argument given, in Lemma 5.21, to show that the set U_α (different α!) stretches to near the top and bottom of the tube about f, breaks down. However we can use an alternative approach: fix $\delta > 0$, and choose $\varepsilon > 0$ such that even when we are distance δ from the top edge of the tube at time T, the smaller tube with radius $\varepsilon(t+1)^\beta$ about $\sqrt{2r}t - \alpha(t+1)^\beta + \gamma(T+1)^\beta - \delta$ fits (for all times $t \geq T$) within the tube of radius L about f. Then by using the spine decompositon and Jensen's inequality as in Proposition 5.15, we can bound the probability of contributing to $Z(\infty)$ away from zero (over all T). We may take the same approach when starting from a position closer to the centre of the tube (that is, further than δ from the edge). Thus, for small enough α', $U_{\alpha'}$ stretches to within δ of the edge of the tube for *all* times $t \geq 0$. The rest of the proof follows as in Lemma 5.22 and Proposition 5.23. □

We saw in our introduction that the asymptotic speed of the right-most particle in a BBM is $\sqrt{2r}$. The theorem above concerns asking particles to stay close to this critical line forever: for example, we might ask particles to be in $(\sqrt{2r}t - 2\alpha t^\beta, \sqrt{2r}t)$ *for all times* $t \geq 0$. If $\beta > 1/3$ then particles manage this with positive probability; if $\beta < 1/3$ then they do not. What if $\beta = 1/3$? Intuitively this question is "even more critical" than the previous theorem. Indeed, our methods are not able to give a full answer, but they can identify regimes where each behaviour (growth or death) is observed.

Theorem 5.28:
Consider the case $\beta = 1/3$. Let

$$\gamma_0 := \left(\frac{3\pi^2}{8\sqrt{2r}}\right)^{1/3} \quad \textit{and} \quad \gamma_1 := \left(\frac{3\pi^2}{4\sqrt{2r}}\right)^{1/3}.$$

If $\gamma < \gamma_0$ and $\alpha < \frac{3\pi^2}{8\gamma^2\sqrt{2r}} - \gamma$, then $\mathbb{P}(\Upsilon = \infty) = 0$; in fact

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{t^{1/3}} \geq \alpha\sqrt{2r} - \frac{3\pi^2}{8\gamma^2} - \gamma\sqrt{2r}$$

and

$$\limsup_{t\to\infty} \frac{\log \mathbb{P}(\hat{N}(t) \neq \emptyset)}{t^{1/3}} \leq \alpha\sqrt{2r} - \frac{3\pi^2}{8\gamma^2} + \gamma\sqrt{2r}.$$

On the other hand, if $\gamma \geq \gamma_1$ and $\alpha > 3\gamma_1/2$, or if $\gamma < \gamma_1$ and $\alpha > \gamma + \frac{3\pi^2}{8\gamma^2\sqrt{2r}}$, then $\mathbb{P}(\Upsilon = \infty) > 0$ and almost surely on survival

$$\liminf_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^{1/3}} \geq \alpha\sqrt{2r} - \frac{3\pi^2}{8(\gamma \vee \gamma_1)^2} - (\gamma \vee \gamma_1)\sqrt{2r}$$

and

$$\limsup_{t\to\infty} \frac{\log |\hat{N}(t)|}{t^{1/3}} \leq \alpha\sqrt{2r} - \frac{3\pi^2}{8\gamma^2} + \gamma\sqrt{2r}.$$

Proof. The first part of the proof proceeds exactly as that of Theorem 5.27, but with

$$\int_0^t (r - \frac{1}{2}f'(s)^2 - \frac{\pi^2}{8L(s)^2} + \frac{L'(s)}{2L(s)})ds = \left(\alpha\sqrt{2r} - \frac{3\pi^2}{8\gamma^2}\right)(t+1)^{1/3} + o(t^{1/3})$$

and

$$E(t) = \gamma\sqrt{2r}(t+1)^{1/3} + o(t^{1/3}) :$$

the spine decomposition converges if

$$-\alpha\sqrt{2r} + \frac{3\pi^2}{8\gamma^2} + \gamma\sqrt{2r} < 0,$$

so $\mathbb{P}(Z(\infty) > 0) > 0$ if

$$\alpha > \gamma + \frac{3\pi^2}{8\gamma^2\sqrt{2r}}.$$

But increasing γ makes the right-hand side of this inequality larger as soon as $\gamma \geq \gamma_1$, and increasing γ can only make $\mathbb{P}(Z(\infty) > 0)$ larger, so (after some rearrangements) we deduce that $\mathbb{P}(Z(\infty) > 0) > 0$ provided *either* $\gamma \geq \gamma_1$ and $\alpha > 3\gamma_1/2$ *or* $\gamma < \gamma_1$ and $\alpha > \gamma + \frac{3\pi^2}{8\gamma^2\sqrt{2r}}$.

Under $\mathbb{Q}$, $Z(t)$ diverges to infinity if $-\alpha\sqrt{2r} + \frac{3\pi^2}{8\gamma^2} - \gamma\sqrt{2r} > 0$. Since $\alpha > 0$, this is impossible if $\gamma \geq \gamma_0$; so we need $\gamma < \gamma_0$ and $\alpha < \frac{3\pi^2}{8\gamma^2\sqrt{2r}} - \gamma$. If $Z(t) \to \infty$ almost surely under $\mathbb{Q}$, then by Lemma 2.8, $Z(t) \to 0$ almost surely under $\mathbb{P}$.

The calculations of the lim infs and lim sups are standard, as in Propositions 5.15, 5.16 and 5.17. However, we must again take a different approach to show that $\{Z(\infty) > 0\}$

agrees with $\{\Upsilon = \infty\}$ up to a set of zero probability. Our proof, below, is specially adapted to this particular case and takes advantage of the convenient — and well-known — fact that $\frac{1}{3} + 2 \times \frac{1}{3} = 1$.

We can easily show, straight from the spine decomposition and as in previous calculations, that for any $\delta \in (0, \gamma/2)$, there exists $\alpha' > 0$ such that $U_{\alpha'}$ stretches to within $\delta t^{1/3}$ of the edges of the tube at time t for any $t > 0$. Thus (in analogy with Lemma 5.22) we would like to show, loosely speaking, that with high probability, particles spend a long time outside the tubes of radius $\delta(s+1)^{1/3}$, $s \in [0,t]$ nested just inside the upper and lower boundaries of our main tube about f. The idea is that if particles do not want to leave $\hat{N}(t)$ then staying near the boundaries of the tube is a bad tactic. To be more precise about this, following the direction of part of the proof of Lemma 5.22 and setting

$$V_s^1 := (f(s) - L(s), f(s) - L(s) + \delta(s+1)^{1/3}) \cup (f(s) + L(s) - \delta(s+1)^{1/3}, f(s) + L(s))$$

and

$$V_s^2 := (-L(s), -L(s) + \delta(s+1)^{1/3}) \cup (L(s) - \delta(s+1)^{1/3}, L(s))$$

we have

$$\begin{aligned}
&\tilde{\mathbb{P}}\Big(\xi_t \in \hat{N}(t), I_\alpha(\xi_t; t) < t/2\Big) \\
&\leq \tilde{\mathbb{P}}\left(\xi_t \in \hat{N}(t), \int_0^t \mathbb{1}_{\{\xi_s \in V_s^1\}} ds > t/2\right) \\
&\leq \tilde{\mathbb{P}}\left[\frac{\mathbb{1}_{\{|\xi_s| < L(s)\ \forall s \in [0,t]\}}}{e^{\int_0^t f'(s) d\xi_s - \frac{1}{2}\int_0^t f'(s)^2 ds}} \mathbb{1}_{\{\int_0^t \mathbb{1}_{\{\xi_s \in V_s^2\}} ds > t/2\}}\right] \\
&\leq e^{-\frac{1}{2}\int_0^t f'(s)^2 ds + |f'(t)| L(t) + \int_0^t |f''(s)| L(s) ds} \\
&\qquad \cdot \tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s \in [0,t],\ \int_0^t \mathbb{1}_{\{\xi_s \in V_s^2\}} ds > t/2\right) \\
&\leq 2e^{-\frac{1}{2}\int_0^t f'(s)^2 ds + |f'(t)| L(t) + \int_0^t |f''(s)| L(s) ds} \\
&\qquad \cdot \tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s \in [0,t],\ \int_0^t \mathbb{1}_{\{\xi_s \in (L(s) - \delta(s+1)^{1/3}, L(s))\}} ds > t/4\right).
\end{aligned}$$

Now, by our calculation of E above, the exponential part

$$2e^{-\frac{1}{2}\int_0^t f'(s)^2 ds + |f'(t)| L(t) + \int_0^t |f''(s)| L(s) ds}$$

is at most $\exp(-rt + \kappa(t+1)^{1/3})$ for some constant κ and all large t. By the many-to-one theorem,

$$\begin{aligned}
&\tilde{\mathbb{P}}\Big(\exists u \in \hat{N}(t) : I_\alpha(u;t) < t/2\Big)\\
&\leq e^{rt}\tilde{\mathbb{P}}\Big(\xi_t \in \hat{N}(t), I_\alpha(\xi_t;t) < t/2\Big)\\
&\leq e^{\kappa(t+1)^{1/3}}\tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s \in [0,t],\ \int_0^t \mathbb{1}_{\{\xi_s \in (L(s)-\delta(s+1)^{1/3}, L(s))\}} ds > t/4\right).
\end{aligned}$$

We attempt to show that, for small $\delta > 0$, the probability

$$\tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s \in [0,t],\ \int_0^t \mathbb{1}_{\{\xi_s \in (L(s)-\delta(s+1)^{1/3}, L(s))\}} ds > t/4\right)$$

is at most $\exp(-2\kappa(t+1)^{1/3})$.

For the sake of brevity we make some approximations here: for example we will use t instead of $t+1$ in various places, and assume throughout that t is large. Let $\tau := \delta^2 t^{2/3}$, define

$$T_0 := \inf\{s > 0 : \xi_s \in (L(s) - \delta(s+1)^{1/3}, L(s))\} \wedge t$$

and for $k \geq 1$ let

$$T_k := \inf\{s > T_{k-1} + \tau : \xi_s \in (L(s) - \delta(s+1)^{1/3}, L(s))\} \wedge t.$$

Then for any $k \geq 0$,

$$\begin{aligned}
\tilde{\mathbb{P}}(|\xi_{T_k+\tau}| < L(T_k+\tau)) &\leq \tilde{\mathbb{P}}(\xi_{T_k+\tau} - \xi_{T_k} < L(T_k+\tau) - L(T_k) + \delta(T_k+1)^{1/3})\\
&= \tilde{\mathbb{P}}(\xi_\tau < \gamma(T_k+\tau+1)^{1/3} - \gamma(T_k+1)^{1/3} + \delta(T_k+1)^{1/3})\\
&\leq \tilde{\mathbb{P}}(\xi_\tau < \gamma(\tau+1)^{1/3} + \delta(t+1)^{1/3})\\
&\approx \tilde{\mathbb{P}}\left(\xi_1 < \frac{\gamma t^{2/9}}{\delta^{1/3} t^{2/3}} + 1\right)
\end{aligned}$$

which is smaller than $\tilde{\mathbb{P}}(\xi_1 < 2)$ when t is large. We now ask how many of the T_k occur strictly before t. We know that if

$$\int_0^t \mathbb{1}_{\{\xi_s \in (L(s)-\delta(s+1)^{1/3}, L(s))\}} ds > t/4$$

then

$$T_0 + \sum_{k \geq 1 : T_{k-1} < t} (T_k - (T_{k-1} + \tau)) \leq \frac{3t}{4}$$

and

$$T_0 + \sum_{k\geq 1: T_{k-1}<t} (T_k - T_{k-1}) \geq t.$$

This tells us that

$$\sum_{k\geq 1: T_{k-1}<t} \tau \geq \frac{t}{4}$$

and hence there must be at least $t/4\tau - 1 = t^{1/3}/4\delta^2 - 1$ of the T_k strictly before t. Let Y be a binomial random variable with parameters $(\lfloor t^{1/3}/4\delta^2 - 2\rfloor, \tilde{\mathbb{P}}(\xi_1 < 2))$. At each T_k, the spine is within distance $\delta(t+1)^{1/3}$ of the boundary of the tube. If it jumps upwards by too much by time $T_k + \tau$, then it leaves the tube; and it has at least $\lfloor t^{1/3}/4\delta^2 - 2\rfloor$ opportunities to do so. Thus we deduce that

$$\tilde{\mathbb{P}}\left(|\xi_s| < L(s)\ \forall s \in [0,t],\ \int_0^t \mathbb{1}_{\{\xi_s\in(L(s)-\delta(s+1)^{1/3}, L(s))\}} ds > t/4\right)$$
$$\leq P(Y=0) \approx (1-\tilde{\mathbb{P}}(\xi<2))^{t^{1/3}/4\delta^2}.$$

By choosing δ small we can make this smaller than $\exp(-2\kappa(t+1)^{1/3})$, which is what we required. The rest of the proof follows just as in Proposition 5.23. □

Theorems 5.27 and 5.28 should be compared with what is currently known about the right-most particle, for example the work of Bramson [4], results on branching Brownian motion with killing, for example Kesten [25], and work on the branching random walk, for example Hu and Shi [19] and Jaffuel [23]. The recent article by Jaffuel [23], in particular, gives results almost analogous to our Theorems 5.27 and 5.28. Kesten [25], if translated into the language of this article, effectively considers a "one-sided" tube with lower boundary the critical line $\sqrt{2r}t$ and no upper boundary — he shows that there is extinction almost surely, and that the probability of survival up to time t decays like $e^{-t^{1/3}}$. Indeed, if we were to consider a tube with lower boundary the line $\sqrt{2r}t$ and upper boundary $\sqrt{2r}t+\alpha t^{1/3}$ we could obtain, by the above methods, a lower bound for Kesten's asymptotic for the probability of survival up to time t, which would agree with Kesten's results up to a constant in the exponent. Unfortunately the corresponding upper bound, and more accurate calculations on the right-most particle in the style of Bramson [4], do not seem to be accessible via our current methods: the error term $E(t)$ outweighs the fine adjustments necessary to investigate such quantities. We hope to carry out further work on these issues in the future.

Bibliography

[1] K. B. Athreya. Change of measures for Markov chains and the $L \log L$ theorem for branching processes. *Bernoulli*, 6(2):323–338, 2000.

[2] J. Bertoin and A. Rouault. Discretization methods for homogeneous fragmentations. *J. London Math. Soc.*, 72(1):91–109, 2005.

[3] J. D. Biggins and A. E. Kyprianou. Measure change in multitype branching. *Adv. in Appl. Probab.*, 36(2):544–581, 2004.

[4] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[5] B. Chauvin and A. Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Related Fields*, 80(2):299–314, 1988.

[6] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[7] J. Engländer and A. E. Kyprianou. Local extinction versus local exponential growth for spatial branching processes. *Ann. Probab.*, 32(1A):78–99, 2004.

[8] Y. Git. Almost sure path properties of branching diffusion processes. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 108–127. Springer, Berlin, 1998.

[9] P. Haccou, P. Jagers, and V. A. Vatutin. *Branching Processes: Variation, Growth, and Extinction of Populations.* Cambridge University Press, Cambridge, UK, 2005.

[10] R. Hardy and S. C. Harris. A conceptual approach to a path result for branching Brownian motion. *Stochastic Process. Appl.*, 116(12):1992–2013, 2006.

[11] R. Hardy and S. C. Harris. A new formulation of the spine approach to branching diffusions. Preprint, `http://arxiv.org/abs/math/0611054`, 2008.

[12] R. Hardy and S. C. Harris. A spine approach to branching diffusions with applications to L^p-convergence of martingales. In *Séminaire de Probabilités, XLII*, volume 1979 of *Lecture Notes in Math.* Springer, Berlin, 2009.

[13] J. W. Harris and S. C. Harris. Survival probabilities for branching Brownian motion with absorption. *Electron. Comm. Probab.*, 12:81–92, 2007.

[14] J. W. Harris and S. C. Harris. Branching brownian motion with an inhomogeneous breeding potential. *Ann. Inst. Henri Poincar Probab. Stat.*, 2009. To appear.

[15] J. W. Harris, S. C. Harris, and A. E. Kyprianou. Further probabilistic analysis of the Fisher-Kolmogorov-Petrovskii-Piscounov equation: one sided travelling-waves. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(1):125–145, 2006.

[16] S. C. Harris and M. I. Roberts. Branching Brownian motion: almost sure growth along unscaled paths. Preprint, `http://arxiv.org/abs/0811.1704`, 2008.

[17] S. C. Harris and M. I. Roberts. Branching Brownian motion: almost sure growth along scaled paths. Preprint, `http://arxiv.org/abs/0906.0291`, 2009.

[18] S. C. Harris and M. I. Roberts. Measure changes with extinction. *Statist. Probab. Lett.*, 79(8):1129–1133, 2009.

[19] Y. Hu and Z. Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.*, 37(2):742–789, 2009.

[20] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes. I. *J. Math. Kyoto Univ.*, 8:233–278, 1968.

[21] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes. II. *J. Math. Kyoto Univ.*, 8:365–410, 1968.

[22] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes. III. *J. Math. Kyoto Univ.*, 9:95–160, 1969.

[23] B. Jaffuel. The critical random barrier for the survival of branching random walk with absorption. 2009. Preprint, `http://arxiv.org/abs/0911.2227`.

[24] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[25] H. Kesten. Branching Brownian motion with absorption. *Stochastic Processes Appl.*, 7(1):9–47, 1978.

[26] D. Kuhlbusch. On weighted branching processes in random environment. *Stochastic Process. Appl.*, 109(1):113–144, 2004.

[27] T. Kurtz, R. Lyons, R. Pemantle, and Y. Peres. A conceptual proof of the Kesten-Stigum theorem for multi-type branching processes. In K. B. Athreya and P. Jagers, editors, *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pages 181–185. Springer, New York, 1997.

[28] A. E. Kyprianou. Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris' probabilistic analysis. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(1):53–72, 2004.

[29] G. F. Lawler. *Introduction to Stochastic Processes.* Chapman & Hall/CRC, New York/Boca Raton, FL, 2006.

[30] Tzong-Yow Lee. Some large-deviation theorems for branching diffusions. *Ann. Probab.*, 20(3):1288–1309, 1992.

[31] R. Lyons. A simple path to Biggins' martingale convergence for branching random walk. In K. B. Athreya and P. Jagers, editors, *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pages 217–221. Springer, New York, 1997.

[32] R. Lyons, R. Pemantle, and Y. Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.*, 23(3):1125–1138, 1995.

[33] R. Lyons and Y. Peres. *Probability on Trees.* In progress. Available online: http://mypage.iu.edu/∼rdlyons/prbtree/book.pdf.

[34] A.A. Novikov. On estimates and the asymptotic behavior of nonexit probabilities of a Wiener process to a moving boundary. *Math. USSR Sbornik*, 38(4):495–505, 1981.

[35] M. Schilder. Some asymptotic formulae for Wiener integrals. *Trans. Amer. Math. Soc.*, 125:63–85, 1966.

[36] S. R. S. Varadhan. *Large deviations and applications*, volume 46 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

[37] D. Williams. *Probability with martingales.* Cambridge University Press, 1991.