General stochastic epidemic models: non-Markovian SIRS model with varying infectivity and gradual loss of immunity

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Objectif

- As has been observed during the Covid 19 pandemic, when an individual recovers from coronavirus infection, the immunity of the individual persists for some period, after which his immunity decays progressively.
- Usually scientists use the SIRS compartmental model to describe this process. This model assumes that once an individual has recovered, his immunity persists for some period, after which the individual immediately becomes susceptible.
- Thus the goal of this presentation is to define a stochastic epidemic model with varying infectivity and with waning immunity. This can be described by the following scheme.

Illustration

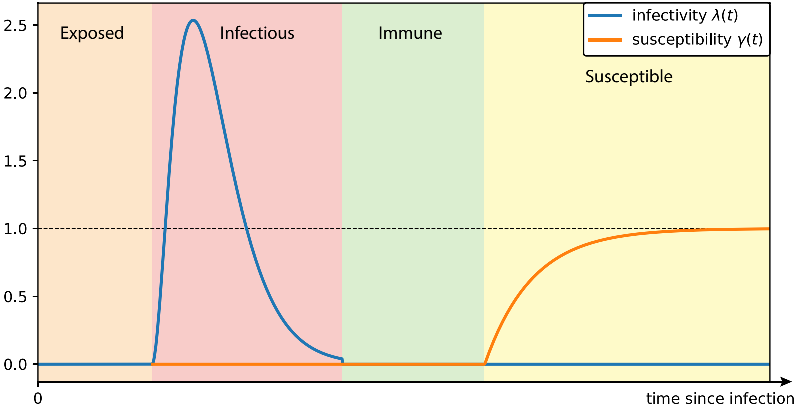

Figure – Evolution of infectivity and susceptibility in time since infection.

Model description

- We are given a fixed size of individual N.
- We assume that some fraction of the population is infected.
- each individual has infectious contacts at a rate equal to its current infectivity.
- At each infectious contact, an individual is chosen uniformly in the population and this individual becomes infected with probability given by its current susceptibility.
- At each infection, a new independent copy of (λ, γ) is drawn to determine the infectivity and susceptibility of the newly infected individual.

Illustration

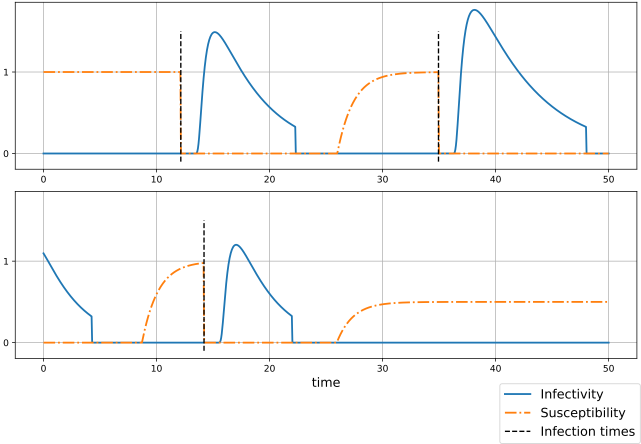

Figure – Illustration of the evolution of an individual's infectivity and susceptibility through time. .

Model description

More precisely, we define

- $\overline{\mathfrak{F}}^N(t)$, the normalised total force of infection at time t, i.e. the sum of the infectivities of all the (infected) individuals at time t, divided by N.
- We also denote the average susceptibility at time t by $\overline{\mathfrak{S}}^N(t)$.

Functional law of large numbers

Theorem 1

Under some basic assumptions,

$$(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N) \xrightarrow[N \to +\infty]{\mathbb{P}} (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}) \text{ in } D^2$$

where $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ solves the set of equation below.

Functional law of large numbers

$$\begin{cases} \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ \qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \\ \overline{\mathfrak{F}}(t) = \bar{I}(0)\bar{\lambda}_0(t) + \int_0^t \bar{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \end{cases}$$

where γ_0 and γ are random functions which represent the susceptibility of the individual before and after the first (re-)infection respectively. And where $\bar{\lambda}_0(t)$ and $\bar{\lambda}(t)$ represent the mean of infectivity of the individual before and after the first (re-)infection respectively.

Illustration

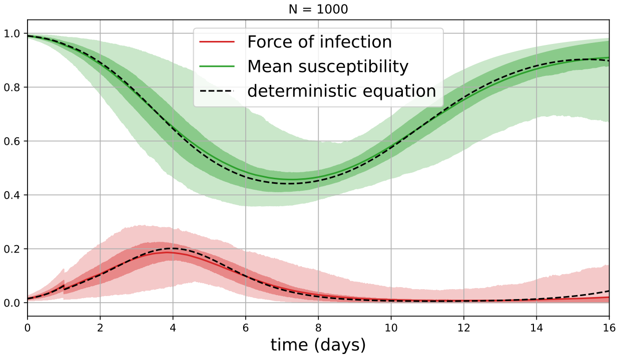

Figure – Stochastic model vs deterministic model.

Global stability of the disease free steady state

we recall that $R_0 = \int_0^{+\infty} \overline{\lambda}(s)ds$.

Theorem 2

When $R_0 < \mathbb{E}\left[\left(\sup_\tau \gamma(\tau)\right)^{-1}\right]$ *and* $R_0 < +\infty$, *as* $t \to +\infty$, $\overline{\mathfrak{F}}(t) \to 0$

Endemic Equilibrium

We assume that the function $t \mapsto \gamma(t)$ is almost surely non-decreasing and we define

$$\gamma_* = \sup_{t \geq 0} \gamma(t) = \lim_{t \to +\infty} \gamma(t).$$

Characterisation of the endemic equilibrium

Theorem 3

Under the Assumptions above and if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. *If there exists* $(\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$ *such that* $(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \xrightarrow[t\to+\infty]{} (\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$, *either* $\overline{\mathfrak{S}}_* \in [0,1]$ *and* $\overline{\mathfrak{F}}_* = 0$, *or else*

$$\overline{\mathfrak{S}}_* = \frac{1}{R_0}$$

and $\overline{\mathfrak{F}}_*$ *is the unique positive solution of the equation*

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\int_0^s \gamma\left(\frac{r}{\overline{\mathfrak{F}}_*}\right) dr\right)\right] ds = R_0. \tag{1}$$

Conjecture

Conjecture 2.1

Under Assumptions, if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ *and* $\overline{\mathfrak{F}}(0) > 0$*, then*

$$(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t)) \to (\overline{\mathfrak{F}}_*, \overline{\mathfrak{S}}_*) \quad \text{as} \quad t \to \infty,$$

where $\overline{\mathfrak{S}}_* = 1/R_0$ *and* $\overline{\mathfrak{F}}_*$ *is the unique positive solution of* (1).

We define $\tau_j x(t) := x(t + t_j)$ where $(t_j)_j \subset \mathbb{R}_+$, $t_j \to +\infty$ as $j \to +\infty$.

By applying the Arzelà-Ascoli Theorem to the set of functions $\{(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}}), j \in \mathbb{N}\}$, there exists a subsequence of pairs $(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}})$ denoted again $(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}})$ such that $(x_j, y_j) := (\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}}) \to (x, y)$ uniformly on compact sets as $j \to +\infty$.

Equivalent conjecture

Note that the pair $(x_j(t), y_j(t))$ satisfies the following system of equations : for $t \geq -t_j$,

$$\begin{cases} x_j(t) = \mathbb{E}\left[\gamma_0(t+t_j)\exp\left(-\int_0^{t+t_j}\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ \qquad + \int_{-t_j}^{t}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)y_j(r)dr\right)\right]x_j(s)y_j(s)ds, \quad (2) \\ y_j(t) = \overline{I}(0)\overline{\lambda}_0(t+t_j) + \int_{-t_j}^{t}\overline{\lambda}(t-s)x_j(s)y_j(s)ds. \quad (3) \end{cases}$$

As a result, as the first terms of the right hand side of (2) and (3) tend to zero when $j \to +\infty$, and $(x_j(t), y_j(t)) \to (x(t), y(t))$ for all $t \in \mathbb{R}$, we deduce by the dominated convergence theorem that the pair (x, y) satisfies the following set of equations,

$$\begin{cases} y(t) = \int_{-\infty}^{t} \overline{\lambda}(t-s)x(s)y(s)ds, & (4) \\ \int_{-\infty}^{t} \mathbb{E}\left[\exp\left(-\int_{s}^{t}\gamma(r-s)y(r)dr\right)\right] x(s)y(s)ds = 1. & (5) \end{cases}$$

We can remark that the constant pair $(\frac{1}{R_0}, \overline{\mathfrak{F}}_*)$ where $\overline{\mathfrak{F}}_*$ is the unique solution of (1) is a solution of (4)-(5). Hence if this solution is unique, all converging subsequences of $(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}})$ have the same limit, from which we can easily conclude the convergence of $(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t))$ as $t \to \infty$.
Thus, Conjecture 2.1 is equivalent to the following.

Conjecture 2.2

Under some Assumptions, if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ and $\overline{\mathfrak{F}}(0) > 0$, the set of equations (4)-(5) has a unique positive and bounded solution on $\mathbb{R}$.

Raphaël Forien, Guodong Pang, Étienne Pardoux, et Arsene Brice Zotsa Ngoufack : Stochastic epidemic models with varying infectivity and susceptibility, arXiv preprint arXiv :2210.04667, 2022.

Thank you for your attention.