Hammond and Sheffield's power law Pólya's urn

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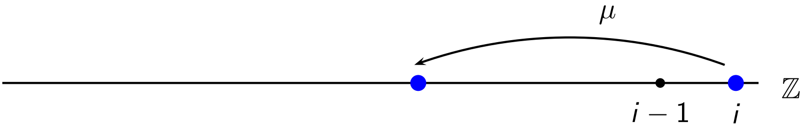
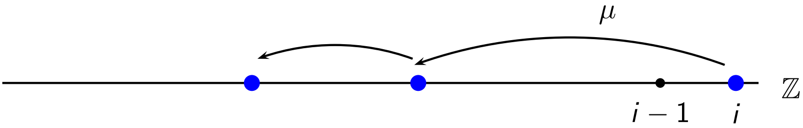
LIFE 5 February 2020

By Alice Klein

Historical accounts describe date palm trees in the Judean desert
Artpartner-images/Getty Images

A simple model...


μ
$\mathbb{Z}$
$i-1$
i


μ
$i-1$
i
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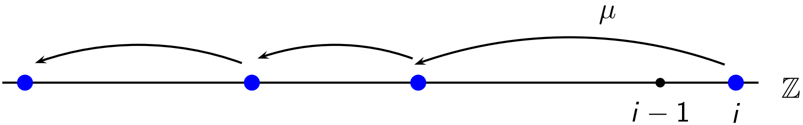
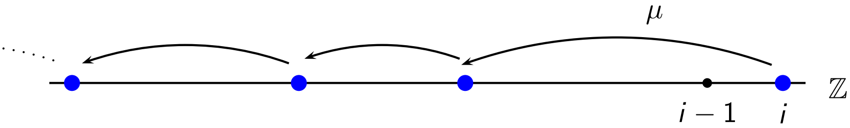
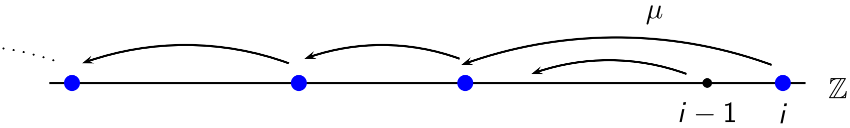

μ
ℤ
i − 1
i


μ
i − 1
i
ℤ



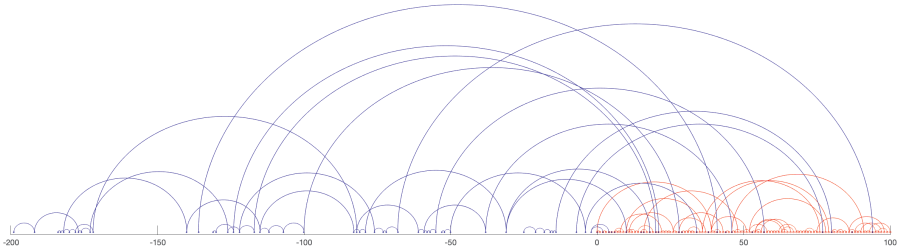
μ
$i-1$
i
$\mathbb{Z}$

$$\mu\left(\{n, n+1, \ldots\}\right) = n^{-\alpha} L(n), \quad \alpha > 0, L \text{ slowly varying.}$$

$$L \text{ slowly varying} :\Leftrightarrow \lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1 \quad \forall a > 0.$$

Just think of $L(n) \equiv 1$.

The random partition G_μ (of $\mathbb{Z}$)



Two regimes

1. $\alpha \in \left(0, \frac{1}{2}\right) \Rightarrow \mathbf{E}\left[R^{\frac{1}{2}}\right] = \infty$ for $R \sim \mu$
2. $\alpha \in \left(\frac{1}{2}, 1\right) \Rightarrow \mathbf{E}\left[R\right] = \infty$ and $\mathbf{E}\left[R^{\frac{1}{2}}\right] < \infty$ for $R \sim \mu$

$\alpha \in \left(0, \frac{1}{2}\right)$

The random partition G_μ of $\mathbb{Z}$ has infinitely many components a.s.!

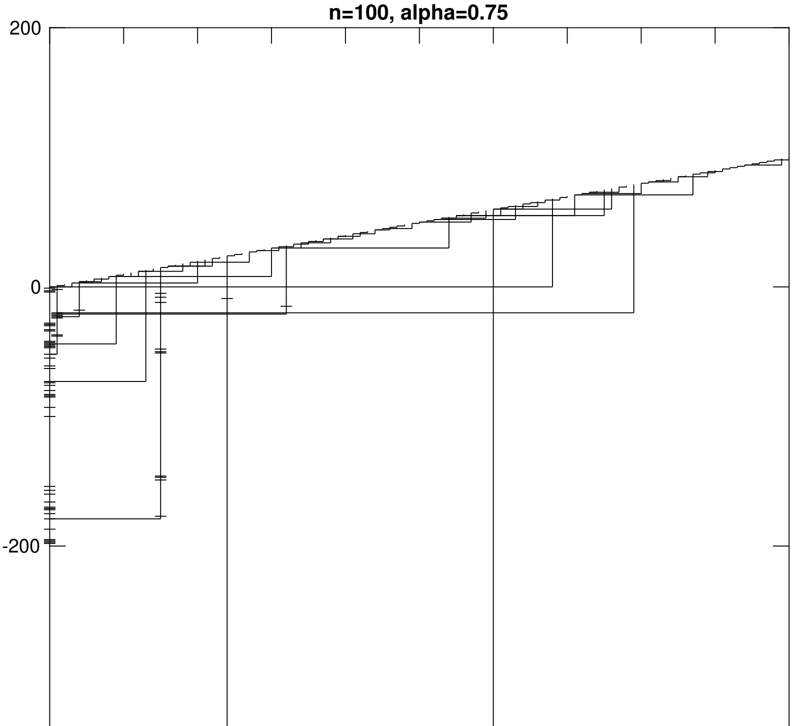
This story is told in Hammond& Sheffield *Power law Pólya's urn and fractional Brownian motion*, PRTF 2013 and taken up in

I.& Wakolbinger *Asymptotic gaussianity via coalescence probabilites in the Hammond-Sheffield urn*, ALEA2023,

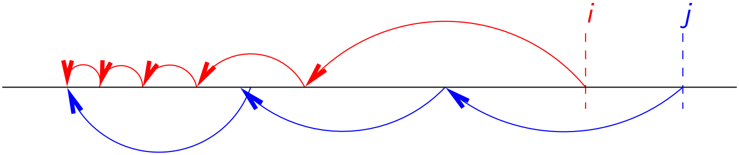
as well as in ongoing work with A. González Casanova *Branching fractional Brownian motion: discrete approximations and maximal displacement*.

$\alpha \in \left(\frac{1}{2}, 1\right)$

The random partition G_μ of $\mathbb{Z}$ has one component a.s.!


n=100, alpha=0.75
200
0
-200

MRCA of two individuals i and j



Distribution of $\mathrm{MRCA}(0, n)$ for $n \to \infty$

$$\frac{\mathrm{MRCA}(0,n)}{n} \overset{(d)}{=} \frac{\mathrm{MRCA}(i,i+n)}{n} \overset{n\to\infty}{\Longrightarrow} \mathrm{Beta}'(1-\alpha, 2\alpha-1)$$

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In particular this means

$$\mathbf{P}\left(\mathrm{MRCA}(0, n)/n > x\right) \sim \mathrm{const} \cdot x^{1-2\alpha}$$

and even more for all sequences $\theta_n \to \infty$

$$\mathbf{P}\left(\mathrm{MRCA}(0, n)/n > x\theta_n\right) \sim \mathrm{const} \cdot (x\theta_n)^{1-2\alpha}.$$

But what about $\mathrm{MRCA}([0, n])$?

Observe first that only $\approx n^{1-\alpha}$ individuals do not coalesce in their first jump.

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Observe first that only $\approx n^{1-\alpha}$ individuals do not coalesce in their first jump.

Open questions:

- How long does it take for $n^{1-\alpha} - n^{1-\varepsilon}$ many individuals to find their MRCA?
- On which scale does $\mathrm{MRCA}\left([0, n]\right)$ typically live?
- Is the game decided by the one individual of $[0, n]$ with the largest jump? This would be of order $n^{\frac{1}{\alpha}}$.