Solutions to Math 263 sheet 6. MR.

① a) $\frac{3^i}{16} = \frac{e^{i\ln 3}}{16} = \left(\frac{1}{16}\right)e^{i(\ln 3)}$ so $r = \frac{1}{16}$, $\theta = \log 3$.

b) $\frac{1}{16}e^{i(\ln 3)} = \frac{1}{16}\cos(\ln 3) + i\frac{1}{16}\sin(\ln 3)$

so $a = \frac{1}{16}\cos(\ln 3)$ and $b = \frac{1}{16}\sin(\ln 3)$.

② a) The homogeneous part is $y'' + 6y' + 9y = 0$.

$P(r) = r^2 + 6r + 9 = (r+3)^2$

so the general solution is $Ae^{-3x} + Bxe^{-3x}$.

Now for a particular solution to $y'' + 6y' + 9y = e^x$,

try $y = ce^x$; $y' = ce^x$, $y'' = ce^x$.

Then $y'' + 6y' + 9y = 16ce^x$ so we need $c = \frac{1}{16}$.

Thus the general solution to $y'' + 6y' + 9y = e^x$ is

$$y = Ae^{-3x} + Bxe^{-3x} + \frac{1}{16}e^x.$$

b) $y'' + 6y' + 9y = e^x$

means $(D+3)^2(y) = e^x$, but $(D+3)^2 = e^{-3x}D^2e^{3x}$

so $e^{-3x}D^2(e^{3x}y) = e^x$

so $D^2(e^{3x}y) = e^{4x}$

so $D(e^{3x}y) = \frac{1}{4}e^{4x} + C_1$

so $e^{3x}y = \frac{1}{16}e^{4x} + C_1x + C_2$

so $$y = \frac{1}{16}e^x + C_1xe^{-3x} + C_2e^{-3x}.$$

(3) Using the substitution $x = e^t$, the equation

$$x^2 y'' + 7xy' + 9y = x$$

becomes $\frac{d^2}{dt^2} y + 6 \frac{d}{dt} y + 9y = e^t$

which (from question 2) has the general solution

$$y = \frac{1}{16} e^t + A e^{-3t} + B t e^{-3t}$$

$$= \frac{x}{16} + \frac{A}{x^3} + \frac{B \ln x}{x^3} .$$

$$y(1) = 0 \Rightarrow 0 = \frac{1}{16} + A \Rightarrow A = -\frac{1}{16}$$

$$y'(x) = \frac{1}{16} + \frac{3}{16x^4} + \frac{B}{x^4} - \frac{3B \ln x}{x^4}$$

$$y'(1) = 0 \Rightarrow 0 = \frac{1}{16} + \frac{3}{16} + B \Rightarrow B = -\frac{1}{4}$$

so $$y = \frac{x}{16} - \frac{1}{16x^3} - \frac{\ln x}{4x^3} .$$