Math 263 Sheet 4 solutions MR.

① a) No. $2(3,2,0)-3(1,0,3)-(3,4,-9)=0$.

(Note also that $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 0 & 3 \\ 3 & 4 & -9 \end{vmatrix} = -36+36 = 0$.)

b) Yes. $\begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 2-3 = -1 \neq 0$.

② a) Yes. For example with $x_1 = 0$, $x_2 = \pi/2$:

$$c_1 \times 1 + c_2 \times 1 = 0$$
$$c_1 \times e^{\pi/2} + c_2 \times 0 = 0 \Rightarrow c_1 = 0$$
$$\Rightarrow c_2 = 0.$$

(Or, $\begin{vmatrix} 1 & 1 \\ e^{\pi/2} & 0 \end{vmatrix} = -e^{\pi/2} \neq 0$.)

b) Yes. For example with $x_0 = 0$, if $y_1 = \cos x$, $y_2 = \cos 2x$ and $y_3 = \sin 3x$ then $y_1' = -\sin x$, $y_2' = -2\sin 2x$, $y_3' = 3\cos 3x$, $y_1'' = -\cos x$, $y_2'' = -4\cos 2x$ and $y_3'' = -9\sin 3x$. So we get

(1) $c_1 \times 1 + c_2 \times 1 + c_3 \times 0 = 0$

(2) $c_1 \times (0) + c_2 \times 0 + c_3 \times 3 = 0 \Rightarrow c_3 = 0$

(3) $c_1 \times (-1) + c_2 \times (-4) + c_3 \times 0 = 0$

$(1)+(3) \Rightarrow -3c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$.

(Or, $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ -1 & -4 & 0 \end{vmatrix} = 12+3 = 15 \neq 0$.)

c) No. $\cos^2 x - \sin^2 x = \cos 2x$, so $(-1)\times\sin^2 x + 1 \times \cos^2 x + (-1)\times \cos 2x = 0$

③ No. Two LI solns to this equation would have to be $y_1 = e^x$ and $y_2 = x^2$. But then we would have

$$W(x) = \begin{vmatrix} e^x & x^2 \\ e^x & 2x \end{vmatrix} = 2xe^x - x^2e^x \; ; \quad \text{so}$$

$$W(1) = 2e - e = e > 0, \quad \text{and } W(-1) = -2e^{-1} + e^{-1} = -e^{-1} < 0.$$

We showed that for solutions to linear homogeneous ODEs, $W(x)$ does not change sign, so y_1 and y_2 cannot be solutions to the same 2nd order linear homogeneous ODE.

④ $\ker(L)$ is the set of points that L maps to 0.

That is, the set of y such that $\frac{d^2y}{dx^2} = 0$.

This is the set of y that look like $Ax+B$ for constants A and B.

$$\ker(L) = \{y : L(y) = 0\}.$$

$$= \{y : \frac{d^2y}{dx^2} = 0\}.$$

$$= \{y : y = Ax+B,\ A,B \in \mathbb{R}\}.$$

⑤ We know $W(x) = W(0)\, e^{-\int_0^x (-a_2(t))dt}$ where $a_2(x) = -x$.

So $W(x) = 1 \times e^{\int_0^x t\,dt} = e^{\frac{t^2}{2}}$.

So $W(\sqrt{2\ln 7}) = e^{\frac{2\ln 7}{2}} = e^{\ln 7} = 7$.