Solutions to MA263 sheet 12. MR.

① $\det(A-\lambda I) = -\lambda\left((-2-\lambda)(7-\lambda)+8\right) - (\lambda-7-8) + 2(-4-4(2+\lambda))$

$$= -\lambda(\lambda^2-5\lambda-6) - \lambda + 15 - 24 - 8\lambda$$
$$= -\lambda^3 + 5\lambda^2 - 3\lambda - 9$$
$$= (\lambda+1)(-\lambda^2+6\lambda-9)$$
$$= -(\lambda+1)(\lambda-3)^2.$$

So the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = 3$.

To find the first eigenvector we set $(A+I)\underset{\sim}{v}_1 = \underset{\sim}{0}$, i.e.

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & -2 \\ -4 & 4 & 8 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives $v_{11} + v_{12} + 2v_{13} = 0$

and $-4v_{11} + 4v_{12} + 8v_{13} = 0$

so $v_{11} = 0$; then we can choose v_{12} (let's choose $v_{12} = 1$) and $v_{13} = -\frac{v_{12}}{2}$.

So $\underset{\sim}{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1/2 \end{pmatrix}$.

For the eigenvector(s) corresponding to $\lambda_2 = 3$, we set

$$\begin{pmatrix} -3 & 1 & 2 \\ -1 & -5 & -2 \\ -4 & 4 & 4 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This time we can choose v_{21} as we like, so let's choose $v_{21} = 1$. Then $-3 + v_{22} + 2v_{23} = 0$

and $-1 - 5v_{22} - 2v_{23} = 0$

so $-4 - 4v_{22} = 0$, so $v_{22} = -1$ and $v_{23} = 2$.

Thus there is ~~~~ only one (LI) eigenvector, $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, corresponding to the eigenvalue $\lambda_2 = 3$.
We look now for solutions of the form

$$\underset{\sim}{x} = \begin{pmatrix}1\\-1\\2\end{pmatrix} t e^{3t} + \begin{pmatrix}a\\b\\c\end{pmatrix} e^{3t}.$$

$$\frac{d\underset{\sim}{x}}{dt} = \begin{pmatrix}1\\-1\\2\end{pmatrix} e^{3t} + \begin{pmatrix}3\\-3\\6\end{pmatrix} t e^{3t} + \begin{pmatrix}3a\\3b\\3c\end{pmatrix} e^{3t}$$

and $$A\underset{\sim}{x} = \begin{pmatrix}3\\-3\\6\end{pmatrix} t e^{3t} + \begin{pmatrix}b+2c\\-a-2b-2c\\-4a+4b+7c\end{pmatrix} e^{3t}$$

So we need
$$b + 2c = 1 + 3a, \qquad (1)$$
$$-a - 2b - 2c = -1 + 3b, \qquad (2)$$
$$-4a + 4b + 7c = 2 + 3c. \qquad (3)$$

We choose $a = 1$, and then (taking $2\times(2) + (3)$) get $b = -1$ and $c = 5/2$.

Thus the general solution is

$$\underset{\sim}{x} = c_1 \begin{pmatrix}0\\1\\-1/2\end{pmatrix} e^{-t} + c_2 \begin{pmatrix}1\\-1\\2\end{pmatrix} e^{3t} + c_3 \left(\begin{pmatrix}1\\-1\\2\end{pmatrix} t e^{3t} + \begin{pmatrix}1\\-1\\5/2\end{pmatrix} e^{3t} \right).$$

2 $\underset{\sim}{x}(0) = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow$
$$c_2 + c_3 = 1$$
$$c_1 - c_2 - c_3 = 0$$
$$-\tfrac{1}{2}c_1 + 2c_2 + \tfrac{5}{2}c_3 = 0$$

$c_2 = 1 - c_3 \Rightarrow c_1 = 1,$
$c_2 = 4, \; c_3 = -3.$

So $$\underset{\sim}{x} = \begin{pmatrix}0\\1\\-1/2\end{pmatrix} e^{-t} + \begin{pmatrix}1\\-1\\1/2\end{pmatrix} e^{3t} - 3\begin{pmatrix}1\\-1\\2\end{pmatrix} t e^{3t}.$$