MA20034. Probability & Random Processes
Example Sheet Eight

1. Suppose $X \in \mathcal{L}^2$ and Y is discrete. Recall that

$$\mathrm{var}(X) := \mathbb{E}(|X - \mathbb{E}(X)|^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

We define the conditional variance of X given Y as

$$0 \leq \mathrm{var}(X|Y) := \mathbb{E}\left(|X - \mathbb{E}(X|Y)|^2 \middle| Y\right)$$

Deduce that $\mathrm{var}(X|Y) = \mathbb{E}(X^2|Y) - (\mathbb{E}(X|Y))^2$.
Using the tower property, deduce that if $X \in \mathcal{L}^2$ then $\mathbb{E}(X|Y) \in \mathcal{L}^2$.
Show that

$$\mathrm{var}(X) = \mathbb{E}(\mathrm{var}(X|Y)) + \mathrm{var}(\mathbb{E}(X|Y)).$$

2. Suppose that X, Y have a joint PDF on $\mathbb{R}^2$ given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

where ρ is a constant in $(-1, 1)$. This is known as the *standard bivariate normal density.*
(a) Recall that $\int_{\mathbb{R}} e^{-(x-\mu)/2\sigma^2} dx = \sqrt{2\pi\sigma^2}$. Calculate the marginal densities

$$f_Y(y) := \int_{x\in\mathbb{R}} f_{X,Y}(x, y)\, dx, \quad f_X(x) := \int_{y\in\mathbb{R}} f_{X,Y}(x, y)\, dy$$

to show that X and Y are $N(0, 1)$ random variables. In particular, note that $\mathbb{E}(X) = 0 = \mathbb{E}(Y)$, $\mathbb{E}(X^2) = 1 = \mathbb{E}(Y^2)$.
(b) Show that the conditional density of X given $Y = y$,

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x, y)}{f_Y(y)} \quad \text{when } f_Y(y) > 0,$$

is that of a $N(\rho y, 1 - \rho^2)$ distribution.
(c) Deduce that $\mathbb{E}(X|Y) = \rho Y$, hence $\mathbb{E}(X|Y) \sim N(0, \rho^2)$.
(d) Show that

$$\mathbb{E}(XY) = \mathbb{E}(Y\mathbb{E}(X|Y)) = \rho.$$

Deduce that $\mathrm{cov}(X, Y) = \rho$.

3. Let X, Y have joint PDF on $\mathbb{R}^2$ given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y} & \text{for } 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate $f_Y(y)$, the marginal PDF of Y, hence show $Y \sim \mathrm{Unif}[0,1]$.
(b) Find $f_{X|Y}(x|y)$, the conditional PDF of X given $Y = y$, for all y such that $f_Y(y) > 0$. Deduce that conditional on $Y = y \in [0,1]$, $X \sim \mathrm{Unif}[0,y]$.
(c) Calculate $\mathbb{E}(X|Y=y)$. Hence, find an expression for the random variable $Z := \mathbb{E}(X|Y)$. What is the distribution of Z? Calculate $\mathbb{E}(Z)$, hence deduce $\mathbb{E}(X)$.
(d)* Show that

$$f(Y) := \mathbb{P}(X^2 + Y^2 \le 1 | Y) = \min\{1, (Y^{-2} - 1)^{1/2}\}.$$

Use the fact that $Y \sim \mathrm{Unif}[0,1]$ to calculate $\mathbb{E}(f(Y))$, hence deduce that $\mathbb{P}(X^2 + Y^2 \le 1) = \log(1 + \sqrt{2})$.
*[**Hint:** To calculate $\mathbb{E}(f(Y))$, either substitute $y = 1/\cosh\theta$ and use integration by parts / hyperbolic trig identities, or substitute $y = \cos\theta$, use integration by parts and "recall" that $\frac{d}{d\theta}(\log(\sec\theta + \tan\theta)) = \sec\theta$ for $0 \le \theta \le \pi/2$.]*

4. ‡ Suppose X and Y have a joint PDF given by

$$f_{X,Y}(x,y) = \frac{1}{y}\exp\left(-y - \frac{x}{y}\right) \quad \text{for} \quad 0 \le x, y < \infty.$$

(a) Calculate the marginal PDF for Y, $f_Y(y)$. What is the distribution of Y?
(b) Deduce the conditional PDF of X given $Y = y$, $f_{X|Y}(x|y)$. Do you recognise this conditional distribution of X given $Y = y$?
(c) Write down the conditional mean, $\mathbb{E}(X|Y=y)$ and conditional variance, $\mathrm{var}(X|Y=y)$, of X given $Y = y$.
(d) Give expressions in terms of Y for the random variables $V := \mathbb{E}(X|Y)$ and $W := \mathrm{var}(X|Y)$. Determine the PDFs for RVs V and W.
(e) Calculate $\mathbb{E}(V)$, hence deduce $\mathbb{E}(X)$.

5. ‡ Suppose that $\Theta \sim \mathrm{Unif}[0,1]$ and, conditional on Θ, $Z_n \sim B(n,\Theta)$.
(a) Let θ be a constant in $[0,1]$. Write down the conditional distribution of Z_n given $\Theta = \theta$, that is, state $\mathbb{P}(Z_n = k|\Theta = \theta)$ for suitable k. Deduce that $\mathbb{E}(Z_n|\Theta = \theta) = n\theta$.
(b) Determine the random variable $\mathbb{E}(Z_n|\Theta)$ and state its distribution. Deduce that $\mathbb{E}(Z_n) = n/2$.
(c) Suppose $k, l \in \mathbb{Z}^+$. Show that

$$I(k,l) := \int_0^1 \theta^k (1-\theta)^l \, d\theta = \frac{k!\, l!}{(k+l+1)!}.$$

*[**Hint:** Show that $I(k,0) = 1/(k+1)$ and $I(k,l) = l\, I(k+1, l-1)/(k+1)$ using integration by parts.]*
(d) Find the marginal distribution for Z_n, that is, calculate

$$\mathbb{P}(Z_n = k) = \int_0^1 \mathbb{P}(\Theta \in d\theta; Z_n = k) = \int_0^1 \mathbb{P}(Z_n = k|\Theta = \theta)\mathbb{P}(\Theta \in d\theta).$$

(e) Show that conditional on $Z_n = k$, $\Theta \sim B(k+1, n-k+1)$, a Beta distribution with parameters $(k+1, n-k+1)$, that is:

$$\mathbb{P}(\Theta \in d\theta | Z_n = k) = \frac{(n+1)!}{k!\,(n-k)!}\theta^k (1-\theta)^{n-k}\, d\theta$$

(f) Calculate $\mathbb{E}(\Theta|Z_n = k)$, hence deduce that

$$\mathbb{E}(\Theta|Z_n) = \frac{Z_n + 1}{n+2}.$$

That is, given Z_n successes in n independent Bernouilli trials each with some probability Θ assumed to be uniformly distributed on $[0,1]$, that Θ should be estimated by $(Z_n+1)/(n+2)$.
Note, this is known as the *rule of succession* and was introduced in the 18$^{\text{th}}$ century by *Pierre-Simon Laplace* to solve the *sunrise problem.*

Note: Questions marked with ‡ are *optional* and * are *harder*.