MA20034. Probability & Random Processes
Example Sheet Six

1. Suppose X is a positive RV. Prove Markov's inequality that

$$c\mathbb{P}(X \geq c) \leq \mathbb{E}(X)$$

Deduce Chebychev's inequality. *[**Hint:** Set $X = (Y - \mu)^2$.]*

2. Suppose a fair die is rolled 144 times and let Y be the total number of sixes obtained. State the distribution of Y. What is the mean and variance of Y? Use Chebychev's inequality to find a 95% confidence interval for Y.

3. Suppose that $X_1, X_2, X_3, \ldots$ are independent RVs with

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^c}, \quad \mathbb{P}(X_n = 1) = \frac{1}{n^c},$$

where $c > 0$ is some constant.
(a) Write down an expression for $\mathbb{P}(|X_n| \leq \epsilon)$ for any $n \in \mathbb{N}$ and $\epsilon > 0$.
(b) We say that X_n converges in probability to X if,

$$\forall \epsilon > 0, \quad \mathbb{P}(|X_n - X| > \epsilon) \to 0 \text{ as } n \to \infty.$$

Show that for any $c > 0$, X_n converges in probability to 0.
(c) We say that X_n converges almost surely to X if,

$$\mathbb{P}(X_n \to X) = 1.$$

Define the event $A_n := \{X_n = 1\}$. Use the Borel-Cantelli Lemmas and the fact that $\sum_n \frac{1}{n^c}$ is finite or infinite according to whether $c > 1$ or $c \leq 1$ respectively, to show that:

(i) If $c \leq 1$, infinitely many of the events $A_1, A_2, \ldots$ occur with probability 1. Deduce that $\mathbb{P}(X_n \to 0) = 0$. Does X_n converge almost surely to anything?

(ii) If $c > 1$, only finitely many of the events $A_1, A_2, \ldots$ occur with probability 1. Deduce that X_n converges almost surely to 0.

(d) It can be shown that if X_n converges almost surely to X, then X_n converges in probability to X. Does the converse hold, that is, does convergence in probability guarantee almost sure convergence?

4. Let $X_1, X_2, \ldots$ be IID RVs each exponentially distributed with rate $\lambda > 0$, that is $\mathbb{P}(X_k \geq x) = e^{-\lambda x}$ for all $x \in \mathbb{R}^+$, $k \geq 1$.
Using integration by parts or otherwise, calculate $\mathbb{E}(X_k)$ and $\mathbb{E}(X_k^2)$. Deduce $\mathrm{var}(X_k)$.
Use Chebychev's inequality to show that, for all $n \in \mathbb{N}$,
$$\mathbb{P}\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{1}{\lambda}\right| > \epsilon\right) \leq \frac{k}{n}$$
for some constant k depending on ϵ and λ that you should identify.
Deduce $n^{-1}\sum_{k=1}^n X_k$ converges in probability to $1/\lambda$, so WLLN holds.
What does the Strong Law of Large Numbers (SLLN) say here?
For $z > 0$, use your bound above to say something about
$$\mathbb{P}\left(\left|\frac{\sum_{i=1}^n X_i - n\lambda^{-1}}{\sqrt{n}}\right| > z\right)$$
(Later, we'll see the Central Limit Theorem improve on this...)

5. ‡** **More on Primes and the Riemann-Zeta function...**
Let $s > 1$. Suppose X and Y are IID $\mathbb{N}$-valued RVs with the Euler(s) distribution:
$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = \frac{n^{-s}}{\zeta(s)}.$$
Show that, for a prime p,
$$\mathbb{P}(\text{there is at least one of } X, Y \text{ which } p \text{ fails to divide}) = 1 - p^{-2s}.$$
Let H be the the highest common factor (greatest common divisor) of X and Y. For p prime, let B_p be the event that 'both X and Y are divisible by p'. How does the event $\cap_{p \text{ prime}}(B_p^c)$ relate to H?
Show that
$$\mathbb{P}(H = 1) = \frac{1}{\zeta(2s)} = \sum_{u,v \text{ coprime}} \frac{u^{-s}v^{-s}}{\zeta(s)^2}$$
where the summation is over all pairs (u, v) in $\mathbb{N}^2$ such that the highest common factor of u and v is 1.
Prove that H has the Euler($2s$) distribution, that is, for $n \in \mathbb{N}$
$$\mathbb{P}(H = n) = \frac{n^{-2s}}{\zeta(2s)}.$$

Note: Questions marked with ‡ are *optional* and * are *harder*.