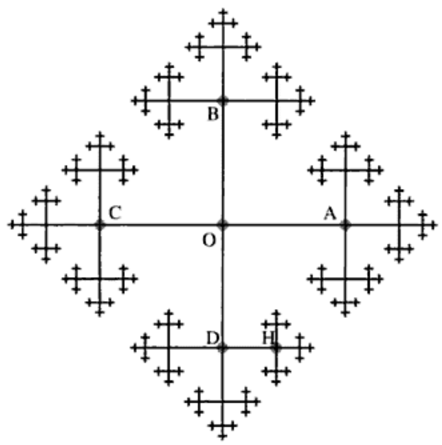
MA20034. Probability & random processes
Example Sheet Four

1. A coin with probability p of Heads and probability $q := 1-p$ of Tails is tossed repeatedly. Let T_n be the number of *Tails* before the n^{th} *Head* is obtained. Define the PGF, $g_{T_n}(\theta) := \mathbb{E}(\theta^{T_n})$.
   By splitting over the outcome of the first toss, show that
   $$g_{T_1}(\theta) = p + q\theta g_{T_1}(\theta).$$
   Prove that
   $$g_{T_n}(\theta) = \left(\frac{p}{1-q\theta}\right)^n$$
   Deduce that if $q = \lambda/n$ then, as $n \to \infty$, $g_{T_n}(\theta) \to \exp(\lambda(\theta-1))$. Interpret this result and explain intuitively.

2. Suppose a coin has probability p of Heads, $q := 1-p$ of Tails. Let V be the number of coin tosses required until you first get the sequence HH. Let $g(\theta) := \mathbb{E}(\theta^V)$.
   (a) By splitting over the events $\{T \text{ on first toss}\}$, $\{HT \text{ on first two tosses}\}$ and $\{HH \text{ on first two tosses}\}$, show that
   $$g(\theta) = \frac{p^2\theta^2}{1-q\theta-pq\theta^2}.$$
   Show that $\mathbb{E}(V) = g'(1) = 1/p + 1/p^2$. In particular, note that for a fair coin, it takes on average 6 tosses to get HH. How many tosses do you guess it takes to get HT on average?
   (b) For the special case when $p = 1/3$, by factorising the denominator and using partial fractions, show that
   $$g(\theta) = \frac{\theta^2}{9}\left(\frac{2}{3}.\frac{1}{(1-\frac{2\theta}{3})} + \frac{1}{3}.\frac{1}{(1+\frac{\theta}{3})}\right)$$
   Hence, by expanding this expression as a power series in θ and equating coefficients, show that
   $$\mathbb{P}(V=n) = \frac{1}{9}\left\{\left(\frac{2}{3}\right)^{n-1} - \left(-\frac{1}{3}\right)^{n-1}\right\} \qquad (n \geq 2).$$

3. Toss a fair coin repeatedly. What is the probability that you get the pattern: (a) HH before HT, (b) TH before HH?
   If I name any one of the patterns HH, HT, TH or TT, which other one would you then choose to maximise the chance that your pattern appears before mine? [***Hint:*** *There is a lot of symmetry.*]

4. * Consider a symmetric RW between neighbouring vertices of the infinite fractal graph whose structure can be indicated pictorially, at least up to the first few levels, by the following picture:



   Any vertex v has four neighbouring vertices and, if the RW is currently at v, it will move to a neighbouring vertex with probability $1/4$ on its next step, independent of the past. The shrinking size of branches is only to permit easy drawing and distances in the picture are not important for the RW, only the neighbourhood structure between vertices is significant. For example, the probability of hitting O starting at A is exactly the same as the probability of hitting D starting at H.
   Prove that

$$\mathbb{P}_{\mathrm{O}}(\text{hit A}) = \frac{1}{3}, \ \mathbb{P}_{\mathrm{O}}(\text{hit A before B}) = \frac{3}{10},$$

$$\mathbb{P}_{\mathrm{O}}(\text{hit both A and B}) = \frac{1}{15},$$

$$\mathbb{P}_{\mathrm{O}}(\text{hit H before either A or B}) = \frac{9}{101}.$$

5. A, B and C throw a fair die in cyclic order (A then B then C then A then...) until one of them wins by getting a 5 or a 6. Let a, b, c be the probabilities that A, B and C win respectively.
(a) Use the 'splitting' technique to find the probabilities a, b, c.
(b) Let T be the duration of the game. Let $g_T(\theta) := \mathbb{E}(\theta^T)$. Show that

$$g_T(\theta) = \frac{\theta}{3 - 2\theta}.$$

(c)* Note the event that A wins is $\cup_{n\geq 0}\{T = 1 + 3n\}$, and write down something similar for the events that B and C win. By multiplying numerator and denominator by $3^2 + 3(2\theta) + (2\theta)^2$, find a, b, c.
Further, find the probabilities that A, B and C win on their n^{th} roll, respectively. Explain intuitively.