Liouville Brownian motion and its heat kernel

Rémi Rhodes

Paris-Bath meeting, june 2014

Joint works with
C.Garban, P.Maillard, V.Vargas, O.Zeitouni

Plan of the talk

1. Gaussian Free Field
2. Liouville Brownian motion
3. Small times asymptotics of the heat kernel

Plan of the talk

1. Gaussian Free Field
2. Liouville Brownian motion
3. Small times asymptotics of the heat kernel

Gaussian Free Field on the $2d$-torus $\mathbb{T}$

Gaussian random distribution (Schwartz) $(X_x)_{x\in\mathbb{T}}$ on D s.t.:

- a.s. X lives in the Sobolev $H^{-1}(D)$
- X is centered and formally $\mathbb{E}[X_x X_y] = G(x,y)$ where G = Green function of Laplacian with vanishing mean

$$-\triangle u = 2\pi f, \quad \int_{\mathbb{T}} u = 0.$$

- short scale divergent behaviour:

$$G(x,y) \sim \ln \frac{1}{d_{\mathbb{T}}(x,y)}, \quad \text{as } d_{\mathbb{T}}(x,y) \to 0.$$

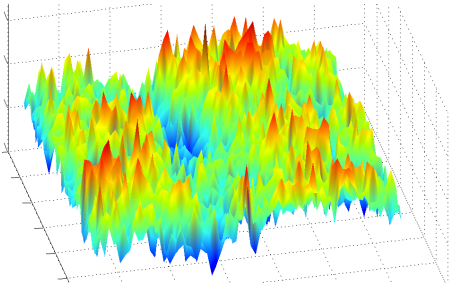
- cannot be defined as a pointwise function

- Let $(\lambda_n)_n$ be the (positive) eigenvalues of $\triangle$ and $(e_n)_n$ the eigenfunctions

$$X(x) = \sum_{k \geq 1} \frac{\alpha_k \, e_k(x)}{\sqrt{\lambda_k}}$$

where $(\alpha_n)_n$ are i.i.d. with law $\mathcal{N}(0,1)$.

- n-th level approximation $\quad X_n(x) = \sum_{k=1}^{n} \frac{\alpha_k \, e_k(x)}{\sqrt{\lambda_k}}$



Plan of the talk

1. Gaussian Free Field
2. Liouville Brownian motion
3. Small times asymptotics of the heat kernel

(critical) Liouville Field Theory (LFT)

Study the "metric tensor" on the torus

$$e^{\gamma X(z)}\, dz^2$$

where X is a Gaussian Free Field and $\gamma \geq 0$ a parameter.

- Motivations coming from $2d$ Liouville quantum gravity
  Refs: Polyakov 81, David 88, Distler-Kawai 88, Duplantier-Sheffield 08,...
- Mathematically not straightforward: $e^{\gamma X(z)}$ is not pointwise defined.
  $\Rightarrow$ renormalization procedure required

Liouville measure (volume form)

How to give sense to the random measure on $\mathbb{T}$?

$$\forall A \subset \mathbb{T}, \quad M_\gamma(A) = \int_A e^{\gamma X(x)}\,dx.$$

Gaussian multiplicative chaos (Kahane 85):

- Cut off the singularity of the field X: use the "smooth" approximations $(X_n)_n$ and define the approximate measures for $\gamma > 0$

$$M_\gamma^n(A) = \int_A e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx.$$

- Positive martingale $\Rightarrow$ almost sure convergence towards a limit $M_\gamma(A)$
- Uniform integrability $\Leftrightarrow \gamma < 2$
- for $\gamma < 2$, M_γ is diffuse and is carried by the γ-thick points, which have Hausdorff dimension $2 - \frac{\gamma^2}{2}$.

(Loading...)

Main goal

Construct the Brownian motion $\mathcal{B}$ of the metric tensor

$$e^{\gamma X(x)}\,dx^2.$$

Formally

$$\mathcal{B}^x_t = B^x_{F^{-1}_\gamma(t)}, \quad F_\gamma(t) = \int_0^t e^{\gamma X(B^x_r)}\,dr.$$

with B standard Brownian motion on $\mathbb{T}$.

- Once again, not straightforward because $e^{\gamma X(z)}$ is not pointwise defined.
  $\Rightarrow$ renormalization procedure required

Fix $x \in \mathbb{T}$. How to give sense to the change of times?

$$F_\gamma(t) = \int_0^t e^{\gamma X(B_r^x)}\, dr.$$

Gaussian multiplicative chaos (again):

- Cut off the singularity of the field X: use the "smooth" approximations $(X_n)_n$ and define the approximation

$$F_\gamma^n(t) = \int_0^t e^{\gamma X_n(B_r^x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(B_r^x)^2]}\, dr.$$

- For each sampling of B^x, it is a positive martingale $\Rightarrow \mathbb{P}^X$ almost sure convergence towards a limit $F_\gamma(t)$
- $\gamma < 2 \Rightarrow$ uniformly integrable and strictly increasing w.r.t t
- Ref: See also N.Berestycki 13'

Question

Show that $\mathbb{P}^X$ almost surely, the change of times F can be defined for all starting points.

- One has to show that, when the environment X is fixed, the law under $\mathbb{P}^{B^x}$ of the change of times F is continuous with respect to x.

Theorem (Garban, R., Vargas 13')

$\mathbb{P}^X$ almost surely,

- one can define the change of times F under $\mathbb{P}^{B^x}$ for all points $x \in \mathbb{T}$.
- under $\mathbb{P}^{B^x}$, F is strictly increasing and continuous.
- the process $\mathcal{B}_t^x = B^x_{F_\gamma^{-1}(t)}$ defines a Feller Markov process with continuous sample paths.

General result of Dellacherie-Meyer for time changes:

$$\mathbb{P}^B\Big(\sup_{s\leq T}|\int_0^s e^{\gamma X(x+B_r)-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,dr-\int_0^s e^{\gamma X(y+B_r)-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,dr|\geq\eta\Big)\leq C\exp\Big(-\frac{\eta}{c\sqrt{g(x,y)}}\Big).$$

with

$$g(x,y)=\sup_{z\in\mathbb{T}}\Big|\int_{\mathbb{T}}G_T(z+y,w)M_\gamma(dw)-\int_{\mathbb{T}}G_T(z+x,w)M_\gamma(dw)\Big|.$$

$$G_T(z,z')=\int_0^T p(r,z,z')\,dr\sim c\ln\frac{1}{|z-z'|}.$$

Multifractal analysis:

there exists C random and $\alpha>0$ such that for all $z\in\mathbb{T}$ and $r<1$

$$M_\gamma\big(B(z,r)\big)\leq Cr^\alpha.$$

Theorem (Garban, R., Vargas 13')

For $\gamma < 2$, a.s. in X,

- the Liouville Brownian motion $\mathcal{B}$ admits the Liouville measure M_γ as unique invariant measure.
- the Liouville semigroup $(P_t^\gamma)_{t\geq 0}$ admits a heat kernel with respect to M_γ

$$P_t^\gamma f(x) = \mathbb{E}^{B^x}[f(\mathcal{B}_t^x)] = \int_{\mathbb{T}} \mathbf{p}_\gamma(t,x,y) f(y) M_\gamma(dy)$$

Consequence: almost surely in X, for all x and t,

$$\mathbb{P}^{B^x}\Big(\mathcal{B}_t \in \{\gamma\text{-thick points}\}\Big) = 1.$$

Theorem (Maillard, R., Vargas, Zeitouni 14')

The heat kernel $\mathbf{p}_\gamma(t,x,y)$ is a continuous function of (t,x,y).

1. the Green function of the LBM is the standard Green function

$$\int_{\mathbb{T}} f\,dM_\gamma = 0 \Rightarrow \int_0^\infty P_t^\gamma f(x)\,dt = \int_{\mathbb{T}} G(x,y) f(y) M_\gamma(dy)$$

2. Consider the eigenfunctions $(e_n)_n$ and eigenvalues $(\lambda_n)_n$ of the Hilbert-Schmidt operator

$$T : f \;\Big(\text{with } \int_{\mathbb{T}} f\,dM_\gamma = 0\Big) \mapsto \int_{\mathbb{T}} G(x,y) f(y) M_\gamma(dy)$$

3. Write

$$\mathbf{p}_\gamma(t,x,y) = \frac{1}{M_\gamma(\mathbb{T})} + \sum_n e^{-\lambda_n t} e_n(x) e_n(y)$$

Heat kernel and fractal properties of Liouville field theory

Main purposes

The heat kernel usually encodes the geometry of the metric tensor. So investigate

- the spectral dimension of LFT (R., Vargas 13')

$$\lim_{t\to 0} -2\frac{\ln \mathbf{p}_\gamma(t,x,x)}{\ln t} = 2$$

- the shape of the heat kernel
- Connection with the Hausdorff dimension of LFT
- an "intrinsic" KPZ formula

1. Gaussian Free Field
2. Liouville Brownian motion
3. Small times asymptotics of the heat kernel

What is the shape of the Liouville heat kernel?

If one believes in the existence of the Liouville distance $\mathbf{d}_\gamma$, one should have

$$M_\gamma(B_\gamma(x,r)) \sim cr^\beta$$

where $B_\gamma(x,r)$ stands for the $\mathbf{d}_\gamma$-balls and β is the Hausdorff dimension of LFT.

Conjecture for short time asymptotics: For x, y fixed and $t \to 0$

$$\mathbf{p}_\gamma(t,x,y) \asymp C(\frac{1}{t}+1)\exp\Big(-c\Big(\frac{\mathbf{d}_\gamma(x,y)^\beta}{t}\Big)^{\frac{1}{\beta-1}}\Big)$$

Remark:
-Well posed problem for studying the Hausdorff dimension β: study the short time asymptotics of the heat kernel

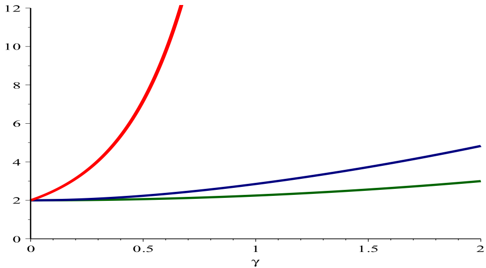
Theorem (Maillard, R., Vargas, Zeitouni 2014)

- An upper bound: for some β_{up} "large" and all $x, y, t, \delta > 0$

$$\mathbf{p}_\gamma(t,x,y) \leq C\Big(\frac{1}{t^{1+\delta}}+1\Big)\exp\Big(-c\Big(\frac{\mathbf{d}_{\mathbb{T}}(x,y)^{\beta_{up}}}{t}\Big)^{\frac{1}{\beta_{up}-1}}\Big).$$

- A lower bound: for all x, y fixed and $\eta > 0$ there exists c random s.t for $t \leq T_0$

$$\mathbf{p}_\gamma(t,x,y) \geq \exp\big(-t^{-\frac{1}{\beta_{low}-1}}\big), \quad \beta_{low} = 2+\frac{\gamma^2}{4}$$



Legend:

Upper bound

Watabiki

Lower bound

The lower bound

- Work with the resolvent density instead and use the bridge formula

$$\mathbf{r}_\lambda(x,y)=\int_0^\infty e^{-\lambda t}\mathbf{p}_\gamma(t,x,y)\,dt=\int_0^\infty \mathbb{E}^{\mathrm{Brid}_t^{x,y}}[e^{-\lambda F(t)}]p(t,x,y)\,dt.$$

- Force the bridge to do atypical things. Trade-off cost/gain on the functional F

$$\mathbb{E}^{\mathrm{Brid}_t^{x,y}}[e^{-\lambda F(t)}]\geq \mathbb{E}^{\mathrm{Brid}_t^{x,y}}[e^{-\lambda F(t)}|A_t]P^{\mathrm{Brid}_t^{x,y}}(A_t)$$

- Force the bridge to stay within a thin tube around $[x,y]$ and speed up the bridge according to the local behaviour of the measure M_γ

- Get the lower bound $\beta_{low}=2+\frac{\gamma^2}{4}$.

General framework (Grigor'yan, Hu, Lau 2010)

Let p_t be the heat kernel of a conservative, local, regular Dirichlet form. Assume for some $\alpha, \beta > 0$

$$p_t(x,y) \leq C(\frac{1}{t^\alpha} + 1), \quad \limsup_{r \to 0} \sup_{y \in \mathbb{T}} \mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) = 0.$$

Then

$$p_t(x,y) \leq c(\frac{1}{t^\alpha} + 1) \exp\Big(- c' \Big(\frac{d_{\mathbb{T}}(x,y)^\beta}{t} \Big)^{\frac{1}{\beta - 1}} \Big).$$

Remark: for the Liouville BM, the spectral dimension tells us $\alpha = 1$.

Estimates of the exit times of balls

- By time change

$$\mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) = \mathbb{P}^y(F(T_{B(y,r)}) \leq r^\beta)$$

where $T_{B(y,r)}$ is the exit time of the standard BM.
- use the negative moments estimates of F to get

$$\mathbb{E}^X\mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) \leq r^{\beta q}\mathbb{E}^X\mathbb{E}^y[F(T_{B(y,r)})^{-q}] = Cr^{\beta q - f(q)}$$

- Use the modulus of continuity of F to extend this estimate over small balls

$$\mathbb{E}^X\big[\sup_{z\in B(y,r^\alpha)} \mathbb{P}^z(\tau_{B(z,r)} \leq r^\beta)\big] \leq Cr^{\beta q - f(q)}$$

- Tile the torus with balls of radius r^α to get

$$\mathbb{E}^X[\sup_{z\in\mathbb{T}} \mathbb{P}^z(\tau_{B(z,r)} \leq r^\beta)] \leq Cr^{-2\alpha+\beta q - f(q)}$$

- Optimize in q and apply Borel-Cantelli to find the best possible β

Thanks!

Plan of the talk

4. Heat kernel based KPZ formula

Consider a set $K \subset \mathbb{T}$ and compute

- its Hausdorff dimension $\dim_\gamma^X(K)$ with the random metric $e^{\gamma X(z)}\,dz^2$.
- its Hausdorff dimension $\dim_e(K)$ with the Euclidian metric dz^2.

What KPZ is all about...

Find a relation between

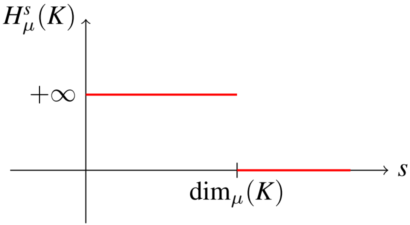
$$\dim_X(K) \quad \text{and} \quad \dim_e(K).$$

- Problem: the construction of the distance associated to $e^{\gamma X(z)}\,dz^2$ remains an open question...

- if one has a distance **d**, one defines the s-dimensional **d**-Hausdorff measure:

$$H^s_{\mathbf{d}}(K) = \lim_{\delta \to 0} \inf \Big\{ \sum_k \mathrm{diam}_{\mathbf{d}}(\mathcal{O}_k)^s;\ K \subset \bigcup_k \mathcal{O}_k, \mathrm{diam}_{\mathbf{d}}(\mathcal{O}_k) \leq \delta \Big\},$$

where the $\mathcal{O}_k$ are open sets.



μ Hausdorff dimension

$$\dim_\mu(K) = \inf\{s \geq 0;\ H^s_{\mathbf{d}}(K) = 0\}.$$

How to measure the dimension of sets?

- if one has a distance **d**, one defines the s-dimensional **d**-Hausdorff measure:

$$H^s_{\mathbf{d}}(K) = \lim_{\delta \to 0} \inf \Big\{ \sum_k \mathrm{diam}_{\mathbf{d}}(\mathcal{O}_k)^s;\ K \subset \bigcup_k \mathcal{O}_k, \mathrm{diam}_{\mathbf{d}}(\mathcal{O}_k) \leq \delta \Big\},$$

where the $\mathcal{O}_k$ are open sets.

- if one has only a measure μ, one defines the s-dimensional μ-Hausdorff measure:

$$H^s_{\mu}(K) = \lim_{\delta \to 0} \inf \Big\{ \sum_k \mu(B_k)^s;\ K \subset \bigcup_k B_k, \mathrm{radius}(B_k) \leq \delta \Big\},$$

where the B_k are closed Euclidean balls.

μ Hausdorff dimension

$$\dim_{\mu}(K) = \inf\{s \geq 0;\ H^s_{\mu}(K) = 0\}.$$

KPZ formula

Fix a compact set K. Consider the Hausdorff dimensions:

- $\dim_{Leb}(K)$ defined with the Lebesgue measure
- $\dim_{M_\gamma}(K)$ defined with the Liouville measure M_γ

Almost surely in X, we have

$$\dim_{Leb}(K) = (1 + \frac{\gamma^2}{4})\dim_{M_\gamma}(K) - \frac{\gamma^2}{4}\dim_{M_\gamma}(K)^2$$

I.Benjamini, O.Schramm: KPZ in one dimensional geometry of multiplicative cascades (2008)

B. Duplantier, S. Sheffield: Liouville Quantum Gravity and KPZ (2008)

R.Rhodes, V.Vargas: KPZ formula for log-infinitely divisible multifractal random measures (2008)

Bauer-David 2009

They object that the latter notion of quantum Hausdorff dimension involves Euclidean balls (not intrinsic to the metric $e^{\gamma X(z)}dz^2$) and suggest a heat kernel formulation of KPZ.

Eulidean capacity dimension:

Fix a set compact set K and define the Euclidean s-capacity of K for $s \in]0,1[$:

$$C_s(K) = \sup\Big\{\Big(\int_{K\times K}\frac{1}{|x-y|^{2s}}\,\mu(dx)\mu(dy)\Big)^{-1}; \mu \text{ Borel}, \mu(K)=1\Big\}$$

and its Euclidean capacity dimension by $\dim_e(K) = \inf\{s \geq 0; C_s(K) = 0\}$

Bauer-David 2009

They object that the latter notion of quantum Hausdorff dimension involves Euclidean balls (not intrinsic to the metric $e^{\gamma X(z)}dz^2$) and suggest a heat kernel formulation of KPZ.

Eulidean capacity dimension:

Fix a set compact set K and define the Euclidean s-capacity of K for $s \in]0,1[$:

$$C_s(K) = \sup\Big\{\Big(\int_{K\times K}\frac{1}{|x-y|^{2s}}\,\mu(dx)\mu(dy)\Big)^{-1}; \mu \text{ Borel}, \mu(K)=1\Big\}$$

and its Euclidean capacity dimension by $\quad \dim_e(K) = \inf\{s \geq 0; C_s(K) = 0\}$

Bauer-David 2009

They object that the latter notion of quantum Hausdorff dimension involves Euclidean balls (not intrinsic to the metric $e^{\gamma X(z)}dz^2$) and suggest a heat kernel formulation of KPZ.

Eulidean capacity dimension:
Fix a set compact set K and define the Euclidean s-capacity of K for $s \in]0,1[$:

$$C_s(K) = \sup\Big\{\Big(\int_{K\times K} \frac{1}{|x-y|^{2s}}\,\mu(dx)\mu(dy)\Big)^{-1}; \mu \text{ Borel}, \mu(K)=1\Big\}$$

and its Euclidean capacity dimension by $\dim_e(K) = \inf\{s \geq 0; C_s(K) = 0\}$

Mellin-Barnes transform of the standard heat kernel:

$$MB(x,y) \overset{def}{=} \int_0^\infty \frac{1}{t^s}\frac{e^{-\frac{|x-y|^2}{2t}}}{2\pi t}\,dt = \frac{c_s}{|x-y|^{2s}}$$

Eulidean capacity dimension:
Fix a set compact set K and define the Euclidean s-capacity of K for $s \in]0,1[$:

$$C_s(K) = \sup\Big\{ \Big(\int_{K\times K} MB(x,y)\,\mu(dx)\mu(dy)\Big)^{-1}; \mu \text{ Borel}, \mu(K)=1\Big\}$$

and its Euclidean capacity dimension by $\dim_e(K) = \inf\{s \geq 0; C_s(K) = 0\}$

Mellin-Barnes transform of the standard heat kernel:

$$MB(x,y) \stackrel{def}{=} \int_0^\infty \frac{1}{t^s} \frac{e^{-\frac{|x-y|^2}{2t}}}{2\pi t}\, dt = \frac{c_s}{|x-y|^{2s}}$$

Eulidean capacity dimension:

Fix a set compact set K and define the Euclidean s-capacity of K for $s \in]0,1[$:

$$C_s(K) = \sup\Big\{\Big(\int_{K\times K} MB(x,y)\,\mu(dx)\mu(dy)\Big)^{-1}; \mu \text{ Borel}, \mu(K)=1\Big\}$$

and its Euclidean capacity dimension by $\dim_e(K) = \inf\{s \geq 0; C_s(K) = 0\}$

Mellin-Barnes transform of the standard heat kernel:

$$MB(x,y) \stackrel{def}{=} \int_0^\infty \frac{1}{t^s}\frac{e^{-\frac{|x-y|^2}{2t}}}{2\pi t}\,dt = \frac{c_s}{|x-y|^{2s}}$$

Quantum capacity dimension:

Define in the same way $\dim_\gamma(K)$ by taking the Melling Barnes transform of the Liouville heat kernel $\mathbf{p}_\gamma(t,x,y)$

Heat kernel based KPZ formula (N.Berestycki, Garban, R. Vargas 14')

Fix a compact set K. Consider the capacity dimensions:

- $\dim_e(K)$ defined with the Mellin Barnes of the Lebesgue heat kernel
- $\dim_\gamma(K)$ defined with the Mellin Barnes of the Lebesgue heat kernel $\mathbf{p}_\gamma$

Almost surely in X, we have

$$\dim_e(K) = (1 + \frac{\gamma^2}{4})\dim_\gamma(K) - \frac{\gamma^2}{4}\dim_\gamma(K)^2$$