Preferential attachment graphs and branching random walks

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joint work with Steffen Dereich (Münster)

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We have analysed a mathematically convenient variant of such graphs. Crucially, the essential features of preferential attachment graphs should not depend on the choice of variant but only on the strength of the preferential attachment and the edge density.

Definition of the model

Take parameters $0 \le \gamma < 1$ and $0 < \beta < 1$ and choose the affine function

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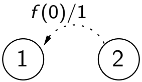

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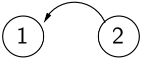

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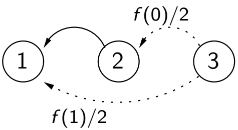

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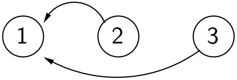

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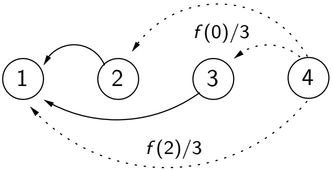

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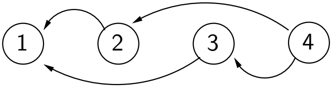

All edges are ordered from the younger to the older vertex. For the questions of interest, edges may be considered as unordered. We denote the resulting increasing sequence of graphs by $(\mathcal{G}_n)$.

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$$\lim_{n\uparrow\infty} X_n(k) = \mu(k) \quad \text{for all } k, \text{ almost surely,}$$

where

$$\mu(k) = \frac{1}{1+f(k)} \prod_{l=0}^{k-1} \frac{f(l)}{1+f(l)} \sim c\, k^{-(1+\frac{1}{\gamma})}.$$

(b) The conditional distribution of the outdegree of the $(n+1)$st vertex, given the graph at time n, converges almost surely in the variational topology to the Poisson distribution with parameter $\sum_{k=0}^{\infty} \mu(k) f(k)$.

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$\Rightarrow$ The empirical degree distribution converges to a power law with exponent

$$\tau := 1 + \frac{1}{\gamma}.$$

Comparison with other models

In the preferential attachment model we have, for $m \leq n$, that

$$\mathbb{E}\big[\text{indegree of } m \text{ at time } n\big] \approx \Big(\frac{n}{m}\Big)^{\gamma},$$

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This makes our model much more difficult to study than rank one models such as

- the LCD-model of Bollobas and Riordan, which is a preferential attachment model that necessarily has $\gamma = \frac{1}{2}$, or, equivalently, $\tau = 3$.
- the configuration model which, loosely speaking, is the uniform distribution on the collection of graphs with a given power law degree distribution.

I. Robustness of networks

Questions:

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(2) When is the network robust?

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Earlier work:

- In the case that $\gamma = 0$ so that there is no preferential attachment the model was first studied by Dubins (1984). The questions were answered by Shepp (1989) and Riordan (2005).
- For the LCD model the questions were answered by Bollobas, Riordan (2003) and Riordan (2005).

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Particle positions are on the real line and types are given by the relative position of their father. Define the pure birth process $(Z_t \colon t \geq 0)$ by its generator

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A particle which has its parent to its left generates offspring

- to its right with relative positions at the jumps of the process $(Z_t : t \geq 0)$;
- to its left with relative positions distributed according to the Poisson process Π on $(-\infty, 0]$ with intensity measure $e^t\, \mathbb{E}[f(Z_{-t})]\, dt$.

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A particle which has its parent to its right generates offspring

- to its right with relative positions at the jumps of $(Z_t^{(\tau)}(t) : t \geq 0)$.
- to its left in the same manner as before.

We start the branching random walk with one initial particle in location $-X$, where X is standard exponential and kill particles and their offspring if their position is to the right of the origin.

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For $0 < \alpha < 1$ and the unkilled branching process we define the matrix $A^{(\alpha)}$ by

$$A^{(\alpha)}_{i,j} := \mathbb{E}_i\Big[\sum_{|x|=1} e^{-\alpha \mathrm{Position}(x)} \mathbf{1}\{\mathrm{Type}(x) = j\}\Big]$$

where $i, j \in \{\text{left}, \text{right}\}$. Denote by $\rho(A^{(\alpha)})$ its spectral radius.

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$$A^{(\alpha)} = \begin{pmatrix} \frac{\beta}{\alpha-\gamma} & \frac{\beta}{1-\alpha-\gamma} \\ \frac{\beta+\gamma}{\alpha-\gamma} & \frac{\beta}{1-\alpha-\gamma} \end{pmatrix}$$

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- The killed branching process dies iff
  there exists $0 < \alpha < 1$ with $\rho(A^{(\alpha)}) \leq 1$.
- The killed branching process has infinite mean growth iff
  the matrix $A^{(\alpha)}$ is ill-defined for any $0 < \alpha < 1$.

Existence of a giant component

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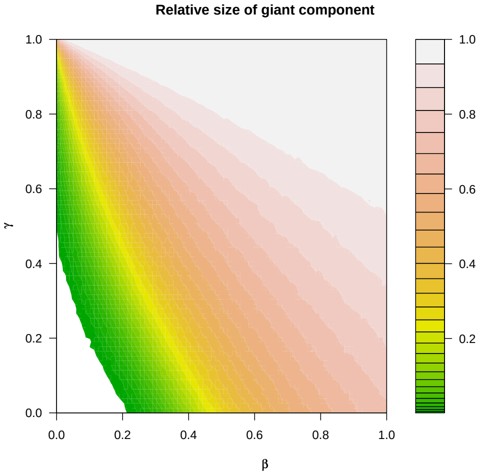
Remark: We also have (slightly less explicit) results in the case that the attachment rule f is concave rather than affine.

Phase diagram



Simulation of the survival probability of the killed branching process.

Size of the giant component near criticality

Denote by $\zeta(\gamma,\beta)$ the size of the giant component which is the asymptotic proportion of vertices in the largest graph component.

Theorem 3 Eckhoff, M (2013)

(a) Let $0 \le \gamma < \frac{1}{2}$ and denote $\beta_c(\gamma) = \frac{(\frac{1}{2}-\gamma)^2}{1-\gamma}$. Then

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(b) Let $0 < \beta < \frac{1}{4}$ and denote $\gamma_c(\beta) = \frac{1}{2}(1 - \beta - \sqrt{\beta^2 + 2\beta})$. Then

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Remark: This is in stark contrast to the behaviour of the configuration model!

Size of the giant component: Idea of proof

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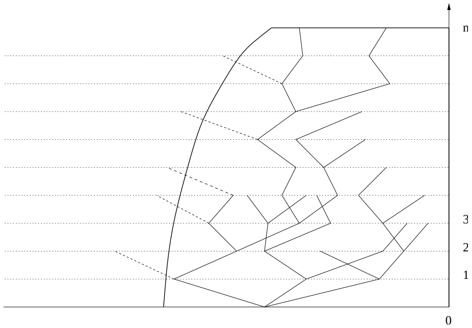

Main ingredients of the proof

- the basis is a truncated first moment method,
- use a many-to-one lemma,
- adapt and use a large deviation theorem for Markov chains.

II. Vulnerability of the network

Besides their robustness under random attack one of the characteristic features of many real networks is their vulnerability to targeted attack. Removal of a small number of key hubs typically changes their behaviour dramatically. We are going to explore this feature in our model of preferential attachment graphs.

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Questions:

- How many old vertices have to be removed to destroy the robustness?
- How does the critical percolation probability behave when the number of removed vertices approaches criticality?
- How are other features like
    - the power law property of the network,
    - the largest degree in the network,
    - the typical distance of vertices in the network,

  affected by the attack?

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We denote by $\mathcal{G}_n^{(\varepsilon)}$ the graph $\mathcal{G}_n$ with vertices $1, 2, \ldots, \varepsilon n$ and adjacent edges removed.

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Let $p_c(\varepsilon)$ be the critical percolation probability of $\mathcal{G}_n^{(\varepsilon)}$.

(a) If $\gamma = \frac{1}{2}$, then as $\varepsilon \downarrow 0$ we have

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Remark: In (a) we conjecture $p_c(\varepsilon) = \frac{1}{\sqrt{\beta(\gamma+\beta)}} \log(1/\varepsilon)^{-1} \big(1 + o(1)\big)$.

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Remark: A similar result holds for the configuration model but the exponent in (b) is doubled from $\gamma - \frac{1}{2}$ to $2\gamma - 1$.

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Now: In the case of two killing boundary at $\log \varepsilon$ and zero, the ancestral line of a surviving particle is confined to the strip $[\log \varepsilon, 0] \times \mathbb{N}$. What is the optimal strategy for a particle to survive in such a strip?

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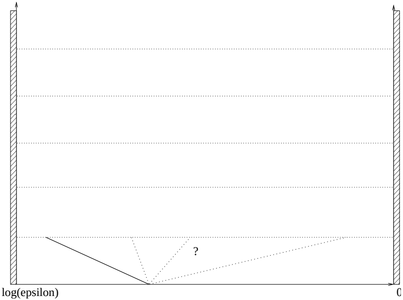

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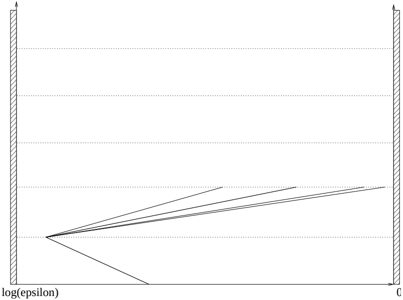

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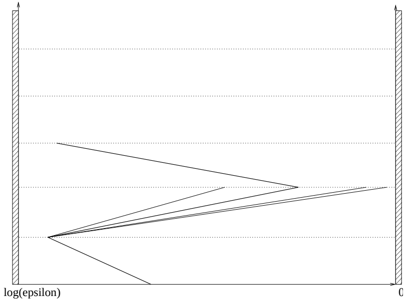

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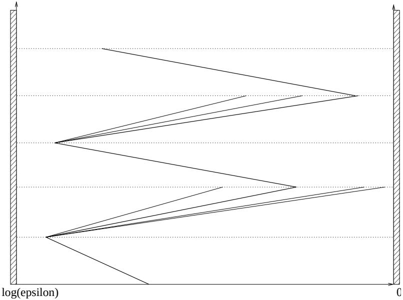

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- for upper bounds guess the eigenfunction ϕ,
- for lower bounds write $A = A_1 + A_2$ for a left offspring operator A_1 and a right offspring operator A_2, and show that main contribution to A^n comes from alternating products of A_1 and A_2.

Loss of the power law

Theorem 6 Eckhoff, M (2012)

Suppose $X_n^{(\varepsilon)}$ is the empirical indegree distribution of $\mathcal{G}_n^{(\varepsilon)}$. Then

$$\lim_{n\uparrow\infty} X_n(k) = \mu^{(\varepsilon)}(k) \quad \text{for all } k, \text{ almost surely,}$$

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Remark: After the attack the network is no longer scale-free. Instead its asymptotic degree distribution has exponential tails.

Increase of typical distances

Preferential attachment graphs with $\gamma > \frac{1}{2}$ are ultrasmall. More precisely, for vertices V_n, W_n chosen uniformly from the giant component of $\mathcal{G}_n$ we have

$$d_{\mathcal{G}_n}(V_n, W_n) \sim \frac{4 \log\log n}{\log \frac{\gamma}{1-\gamma}} \qquad \text{in probability,}$$

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Suppose that $\varepsilon > 0$ is sufficiently small so that $\mathcal{G}_n^{(\varepsilon)}$ has a giant component. and let V_n, W_n be independent, uniformly chosen vertices in $\mathcal{G}_n$. Then, for all $\delta > 0$,

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Remarks: After the attack the network is no longer ultrasmall, and typical distances are now logarithmic, i.e. much larger. The lower bound is conjectured to be sharp.

Summary

- Preferential attachment graphs are built dynamically using a simple and universal principle. From this principle many properties of complex real world networks emerge.

  In particular preferential attachment graphs are scale free, and if the power-law exponent is in the range $2 < \tau < 3$, they are

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Thank you very much for your attention!