TECHNISCHE UNIVERSITÄT DARMSTADT

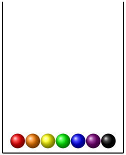
Solutions to multivariate smoothing equations

based on joint work with S. Mentemeier

Example (Cyclic Pólya urns)

Consider an urn with balls of m types (colors) together with the rule that if a ball of type k is drawn, then it is placed back into the urn together with a ball of type $k+1$ (mod m). Then the urn is called cyclic Pólya urn.

Cyclic Pólya urn



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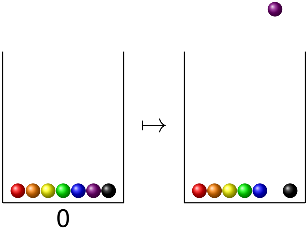

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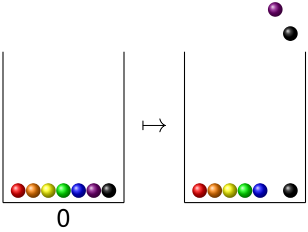

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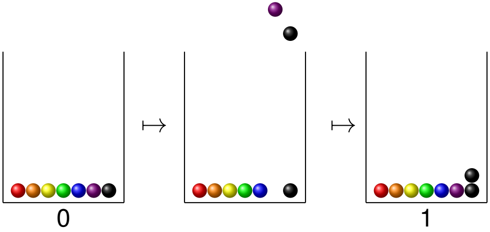

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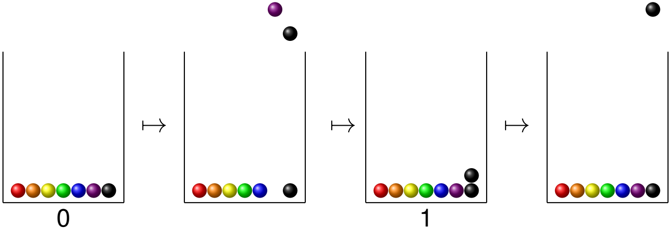

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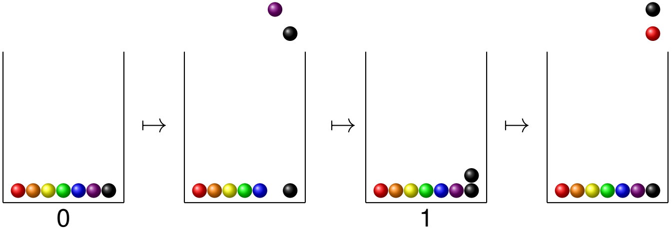

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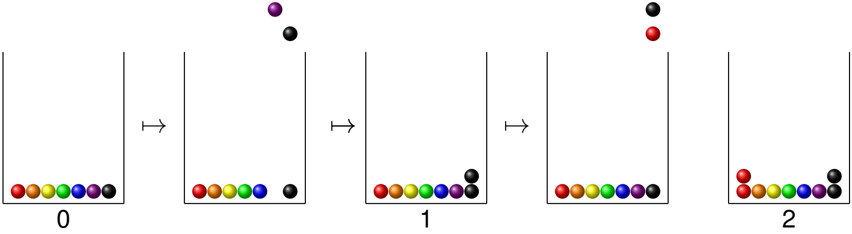

Introduction

Considering the number $R_n^{(k)}$ of balls of type 1 in the urn after n steps when starting with one ball of type k, an explicit formula for $\mathbf{E}[R_n^{(k)}] = \frac{n}{m} + O(1)$ has been derived by Janson (2004).
Now let $m \geq 7$, $\zeta = \exp(2\pi i/m)$ be a primitive mth root of unity with real part $\xi = \cos(2\pi/m)$.
Knape and Neininger (2013) showed that

$$n^{-\xi}(R_n^{(k)} - n/m) \overset{\text{law}}{\approx} \text{Re}\left(e^{i(\sin(\frac{2\pi}{m})\log(n)+2\pi\frac{k-1}{m})}X\right)$$

where X is the unique solution with finite second moment of the equation:

$$X \overset{\text{law}}{=} U^\zeta X_1 + \zeta(1-U)^\zeta X_2 \tag{1}$$

where X_1, X_2 are i.i.d. copies of X that are independent of U which has the uniform distribution on $[0,1]$.

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Example (Asymptotic size of fragmentation trees)

- An object of mass $x = 1$, say, is split into b parts with respective masses $0 \leq V_1, \dots, V_b < 1$ where $b \geq 2$ is a fixed integer and $V_1, \dots, V_b$ are random variables with $V_1 + \dots + V_b = 1$ a.s.
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The fragmentation tree of all objects that have mass strictly $\geq \epsilon$

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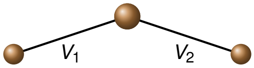

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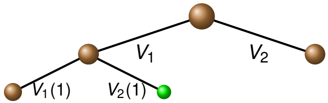

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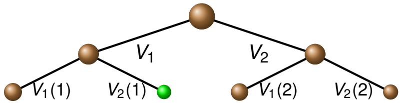

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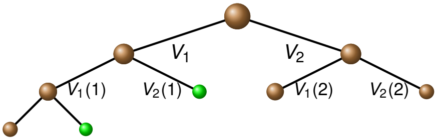

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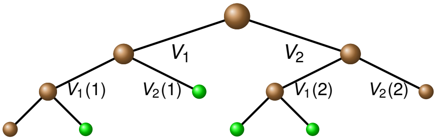

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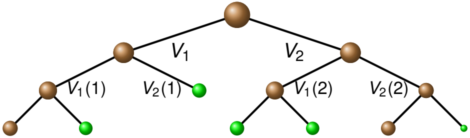

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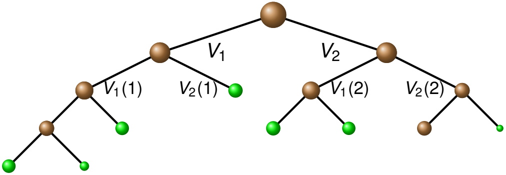

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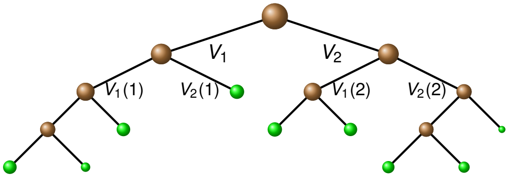

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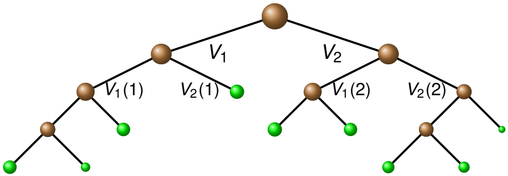

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Introduction

- Let $\psi : \mathbb{C} \to \mathbb{C}$, $z \mapsto \mathbf{E}[\sum_{j=1}^{b} V_j^z]$;
- denote by $1 = \lambda_1, \lambda_2, \lambda_3, ...$ the roots of the equation $\psi(z) = 1$; with the convention that $1 = \mathrm{Re}(\lambda_1) > \mathrm{Re}(\lambda_2) \geq \mathrm{Re}(\lambda_3) \geq ....$

Janson and Neininger (2008) showed that, when $\mathrm{Re}(\lambda_2) \leq 1/2$, $N(\epsilon)$ suitably shifted and scaled, converges in distribution to a centered normal.

On the other hand, when $\mathrm{Re}(\lambda_2) > 1/2$, $N(\epsilon)$ does not converge in distribution. Instead, $\epsilon^{\mathrm{Re}(\lambda_2)}(N(\epsilon) - c\epsilon^{-1})$ has an asymptotic periodic behavior involving the real part of rotations of a complex-valued random variable X with finite second moment and

$$X \stackrel{\text{law}}{=} \sum_{j=1}^{b} V_j^{\lambda_2} X_j \qquad (2)$$

where $X_1, ..., X_b$ are i.i.d. copies of X independent of $(V_1, ..., V_b)$.

Example (m-ary search trees)

Let T_n be an m-ary search tree with n keys inserted.
When $m > 26$, an important role in the asymptotics of T_n is played by a complex-valued solutions W to

$$W \stackrel{\text{law}}{=} e^{-\lambda_2 T}(W_1 + ... + W_m) \tag{3}$$

and

$$W \stackrel{\text{law}}{=} \sum_{j=1}^{m} V_j^{\lambda_2} W_j \tag{4}$$

where $W_1, W_2, ...$ are i.i.d. copies of W independent of T and $(V_1, ..., V_m)$, resp., and $T \sim \tau_1 + ... + \tau_{m-1}$ with independent $\tau_1, ..., \tau_{m-1}$ and τ_j having exponential distribution with parameter j and the V_j are the spacings of $m-1$ independent uniform $(0, 1)$ variables (for details, see Fill and Kapur (2004), Janson (2004), Chauvin, Liu and Pouyanne (2011)).

Example (Kac caricature)

Bassetti and Matthes (2014) considered a generalization of the Kac caricature of the Boltzmann equation which describes the particle velocities V as a vector in $\mathbb{R}^3$. The stationary solution of this equation satisfies

$$V \stackrel{\text{law}}{=} LV_1 + RV_2, \tag{5}$$

where V, V_1, V_2 are i.i.d. and independent of the random pair (L, R) of similarities, which satisfies $\mathbf{E}\left[\|L\|^2 + \|R\|^2\right] = 1$.

General framework:
Multivariate smoothing equations

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Fix $d \in \mathbb{N}$. Let

- $(\mathbf{C}, T_1, T_2, ...)$ be a given sequence of random variables where
  - $\mathbf{C} = (C_1, \dots, C_d)$ is a d-dimensional random vector,
  - $T_1, T_2, ...$ are similarities (*i.e.*, $T_j = \|T_j\| O_j$ for an orthogonal $d \times d$ matrix O_j),
  - and $N := \#\{j \in \mathbb{N} : T_j \neq 0\} < \infty$ a.s.;
- $(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ...)$ be a sequence of i.i.d. $\mathbb{R}^d$-valued random vectors independent of $(\mathbf{C}, T_1, T_2, ...)$.

We consider *multivariate smoothing equations* of the form

$$\mathbf{X} \overset{\text{law}}{=} \sum_{j \geq 1} T_j \mathbf{X}^{(j)} + \mathbf{C}. \tag{6}$$

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Consider $(\mathbf{C}, T_1, T_2, \ldots)$ as given and (the distribution of) $\mathbf{X}$ as unknown.

For which distributions of $\mathbf{X}$ does (6) hold?

Assumptions

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In order to avoid trivial and simple cases as well as case distinctions, we assume throughout that

$$\mathbf{P}\big(\|T_j\| \in r^{\mathbb{Z}} \cup \{0\} \text{ for all } j \geq 1\big) < 1 \text{ for all } r > 1. \tag{A1}$$

$$\mathbf{E}[N] = \mathbf{E}\left[\sum_{j\geq 1} \mathbb{1}_{\{\|T_j\|>0\}}\right] > 1. \tag{A2}$$

The function m

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Let

$$m : [0, \infty) \to [0, \infty], \quad s \mapsto \mathbf{E}\left[\sum_{j=1}^{N} \|T_j\|^s \right].$$

We make the following assumptions:

$$\text{There is an } \alpha > 0 \text{ such that } m(\alpha) = 1. \tag{A3}$$

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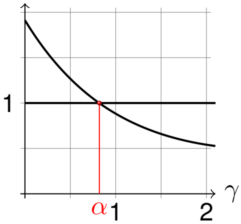
$$\mathbf{E}\left[\sum_{j\geq 1} \|T_j\|^\alpha(-\log\|T_j\|)\right] \in (0, \infty) \quad \text{and} \quad \mathbf{E}[W_1 \log^+ W_1] < \infty. \tag{A4}$$

m and the characteristic exponent



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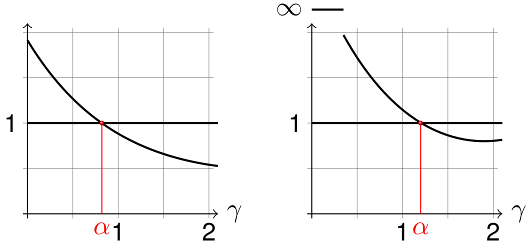

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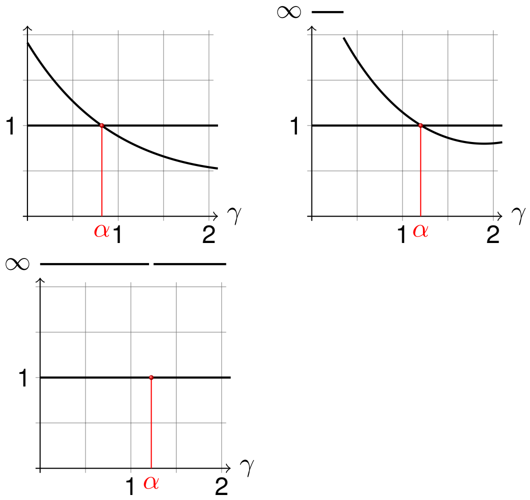

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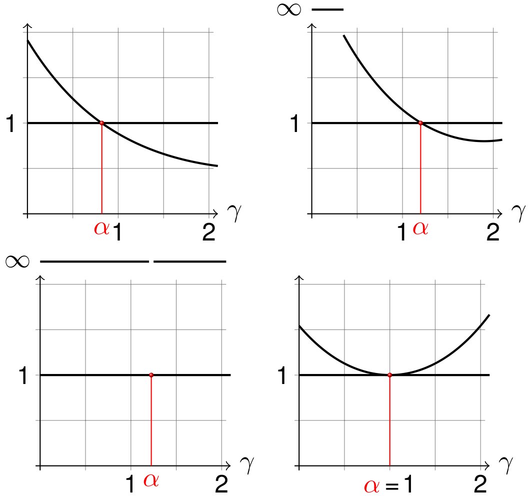
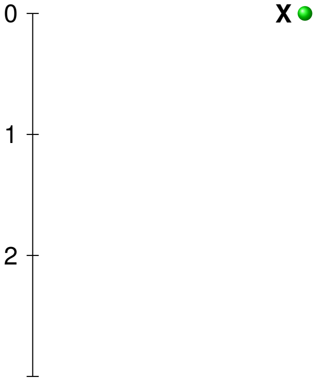


Solving (6): Construction of solutions

Iteration of T_Σ



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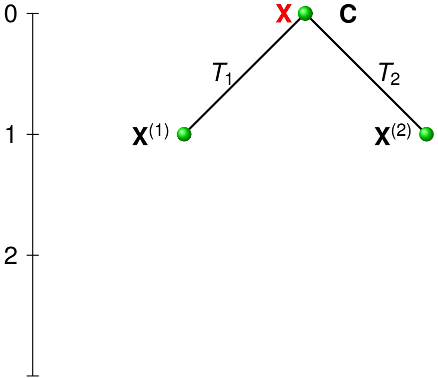

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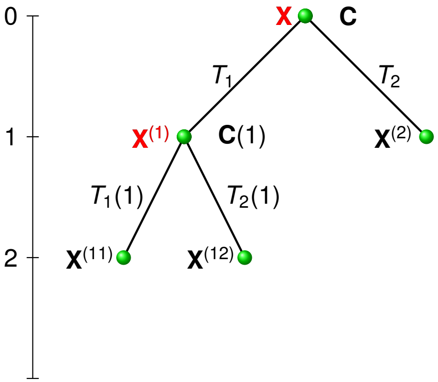

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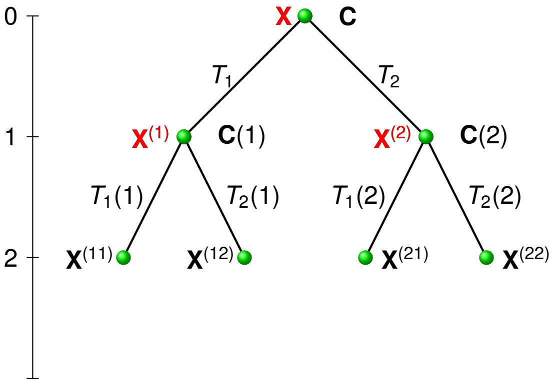

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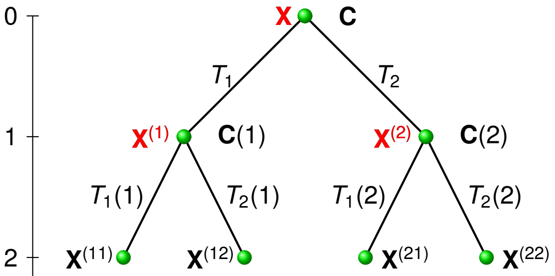

$$\begin{aligned} &\mathbf{C} \\ &+ T_1\mathbf{C}(1) + T_2\mathbf{C}(2) \\ &+ T_1 T_1(1)\mathbf{X}^{(11)} + T_1 T_2(1)\mathbf{X}^{(12)} + T_2 T_1(2)\mathbf{X}^{(21)} + T_2 T_2(2)\mathbf{X}^{(22)} \end{aligned}$$

The weighted branching process

- Let $\mathbb{V} := \bigcup_{n\geq 0} \mathbb{N}^n$ denote the infinite Ulam-Harris tree.
- Let $(\mathbf{C}(v), T(v))_{v\in\mathbb{V}}$ be a family of independent copies of $(\mathbf{C}, T_1, T_2, \ldots)$,

$$\begin{aligned}(\mathbf{C}(v), T(v)) &= (C_1(v), \ldots, C_d(v), T_1(v), \ldots) \\ &\stackrel{\text{law}}{=} (C_1, \ldots, C_d, T_1, T_2, \ldots).\end{aligned}$$

- Let
$$L(\varnothing) := 1 \quad \text{and} \quad L(vj) := L(v)T_j(v), \qquad v \in \mathbb{V},\ j \in \mathbb{N}.$$
- Let $(\mathbf{X}^{(v)})_{v\in\mathbb{V}}$ be a sequence of i.i.d. copies of $\mathbf{X}$ independent of $(\mathbf{C}(v), T(v))_{v\in\mathbb{V}}$.

Constructing a solution to (6)

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$$\mathbf{W}_n^* := \sum_{|v|<n} L(v)\mathbf{C}(v), \quad n \geq 0. \tag{7}$$

By construction, $\mathbf{W}_n^* \overset{\text{law}}{=} T_\Sigma^n(\delta_\mathbf{0})$. Let

$$\mathbf{W}^* := \lim_{n\to\infty} \mathbf{W}_n^* = \sum_{n\geq 0} \sum_{|v|=n} L(v)\mathbf{C}(v) \tag{8}$$

whenever the limit exists in the sense of convergence in probability.
If it does, then $\mathbf{W}^*$ defines a solution to (6).

Constructing homogeneous solutions, II

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By (A3)

$$W_n := \sum_{|v|=n} \|L(v)\|^{\alpha}, \quad n \in \mathbb{N}_0 \tag{9}$$

defines a nonnegative mean-one martingale. We denote its a.s. limit by W. It is well known that $\mathbf{P}(W > 0) > 0$ iff $\mathbf{E}[W] = 1$ and that a sufficient condition for the latter is (A4).

Assume that $T_j \geq 0$, $j \in \mathbb{N}$. Let $\mathbf{Y}, \mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \ldots$ denote a sequence of i.i.d. strictly α-stable random vectors independent of $(\mathbf{C}(v), T(v))_{v \in \mathbb{V}}$.

Define $\mathbf{X} := W^{1/\alpha}\mathbf{Y}$ and $\mathbf{X}^{(j)} := [W]_j^{1/\alpha}\mathbf{Y}^{(j)}$, $j \geq 1$ where $[\cdot]_v$ is the shift by vertex v.

Constructing homogeneous solutions, II

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By (A3)

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$$\begin{aligned}
\sum_{j=1}^{N} T_j \mathbf{X}^{(j)} \;&=\; \sum_{j=1}^{N} T_j([W]_j^{1/\alpha}\mathbf{Y}^{(j)}) \;\overset{\text{law}}{=}\; \left(\sum_{j=1}^{N} T_j^{\alpha}[W]_j\right)^{1/\alpha} \mathbf{Y} \\
&=\; W^{1/\alpha}\mathbf{Y} \;=\; \mathbf{X}.
\end{aligned}$$

The set of solutions in the case of nonnegative weights

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Theorem (Nonnegative weights, Alsmeyer and M. '13)

Suppose that (A1)-(A3) *hold (and some technical condition) and that* $T_j \geq 0$ *a.s. for all* $j \in \mathbb{N}$*. A distribution* P *on* $\mathbb{R}^d$ *is a solution to* (6) *if and only if it is the law of a random variable of the form*

$$\mathbf{W}^* + W^{1/\alpha}\mathbf{Y}_\alpha \tag{10}$$

where

- $\mathbf{W}^*$ *is the special (endogenous) solution to* (6)*;*
- W *is the unique (endogenous) solution to* $W \stackrel{\text{law}}{=} \sum_{j\geq 1} |T_j|^\alpha W_j$*;*
- $\mathbf{Y}_\alpha$ *is strictly* α*-stable and independent of* $(\mathbf{W}^*, W)$*.*

$$\mathbf{X} \stackrel{\text{law}}{=} \sum_{j\geq 1} T_j \mathbf{X}^{(j)} + \mathbf{C}. \tag{6}$$

Constructing homogeneous solutions, III

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Assume that T_j are similarities. Let $\mathbf{Y}$, $\mathbf{Y}^{(1)}$, $\mathbf{Y}^{(2)}$, ... denote a sequence of i.i.d. $\mathbb{G}(O)$-invariant α-stable random vectors independent of $(\mathbf{C}(v), T(v))_{v \in \mathbb{V}}$.

Define $\mathbf{X} := W^{1/\alpha}\mathbf{Y}$ and $\mathbf{X}^{(j)} := [W]_j^{1/\alpha}\mathbf{Y}^{(j)}$, $j \geq 1$.

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Define $\mathbf{X} := W^{1/\alpha}\mathbf{Y}$ and $\mathbf{X}^{(j)} := [W]_j^{1/\alpha}\mathbf{Y}^{(j)}$, $j \geq 1$.

Then

$$\sum_{j=1}^{N} T_j \mathbf{X}^{(j)} = \sum_{j=1}^{N} \|T_j\| (W_j^{1/\alpha}\mathbf{Y}^{(j)}) \overset{\text{law}}{=} \left(\sum_{j=1}^{N} \|T_j\|^{\alpha} W_j \right)^{1/\alpha} \mathbf{Y}$$

$$= W^{1/\alpha}\mathbf{Y} = \mathbf{X}.$$

Determining the set of all solutions

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Is it reasonable to conjecture that in the general case, all solutions are of the form

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for $\mathbb{G}(O)$-invariant α-stable random variables $\mathbf{Y}_\alpha$?

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for $\mathbb{G}(O)$-invariant α-stable random variables $\mathbf{Y}_\alpha$?

Assume $\alpha > 1$ and $\sum_{j\geq 1} T_j = 1$ a.s.
Then $a = \sum_{j\geq 1} T_j a$ a.s., hence adding a constant to (10) gives an additional solution.

Another martingale

In general, we have to take care when $\alpha > 1$ and $\mathbf{E}\left[\sum_{j\geq 1} T_j\right]$ has eigenvalue 1:

- Let $Z_n := \sum_{|v|=n} L(v)w$ where w is an eigenvector to the eigenvalue 1.
- $(Z_n)_{n\geq 0}$ is a martingale.

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- If $(Z_n)_n$ does not converge a.s. or if $Z_n \to 0$ a.s., no further solutions appear.
- If $Z_n \to Z$ a.s. with $\mathbf{P}(Z = 0) < 1$, then aZ, $a \in \mathbb{R}$ are further solutions.

When does $(Z_n)_{n\geq 0}$ converge a.s.?

Martingale convergence

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Lemma

Assume (A1)-(A3)*,* $\mathbf{E}[Z_1] = 1$ *and a technical assumption. Then the following assertions are equivalent:*

(i) $Z_n \to Z$ *a.s.*

(ii) $(Z_n)_{n \geq 0}$ *is bounded in* $\mathcal{L}^\beta$ *for some* $1 < \beta < \alpha$*.*

$\mathcal{L}^\beta$-boundedness is easier to check.
For instance, if $\alpha \geq 2$ (plus an additional technical condition when $\alpha = 2$), $Z_n \to Z$ iff $Z_1 = 1$ a.s.

The set of solutions in the general case

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Conjecture (General case, M. and Mentemeier '14)

Suppose that (A1)-(A3) hold (and some technical condition). A distribution P on $\mathbb{R}^d$ is a solution to (6) if and only if it is the law of a random variable of the form

$$\mathbf{W}^* + \mathbf{a}Z + W^{1/\alpha}\mathbf{Y}_\alpha \tag{11}$$

where

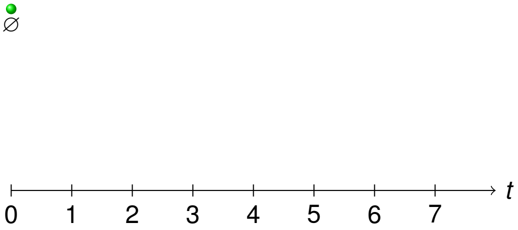
- $\mathbf{W}^*$ and W are above;
- $Z = \lim \sum_{|v|=n} L(v)w$ where w is an eigenvector to the eigenvalue 1 of $\mathbf{E}[\sum_{j\geq 1} T_j]$.
- $\mathbf{Y}_\alpha$ is strictly α-stable and independent of $(\mathbf{W}^*, W, Z)$ and invariant mod $\mathbb{G}(O)$ where $\mathbb{G}(O)$ is the smallest closed multiplicative subgroup of the group of orthogonal matrices generated by the $O_j = T_j/\|T_j\|$, $j \geq 1$.

$$\mathbf{X} \overset{\text{law}}{=} \sum_{j\geq 1} T_j\mathbf{X}_j + \mathbf{C}. \tag{6}$$

Branching processes

The family $(L(v))_{v\in\mathbb{V}}$ can be considered as a multi-type branching process with birth times $S(v) := -\log\|L(v)\|$ and types $O(v) := \|L(v)\|^{-1}L(v) \in \mathcal{O}(d)$. The type space is $\mathbb{G}(O) \subseteq \mathcal{O}(d)$.

Multitype general branching process



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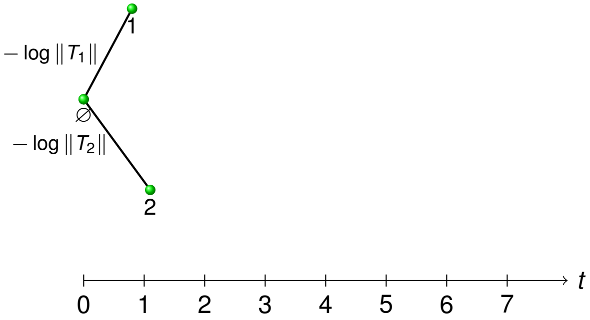

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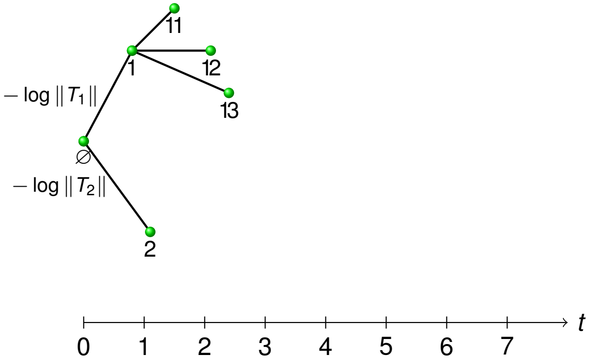

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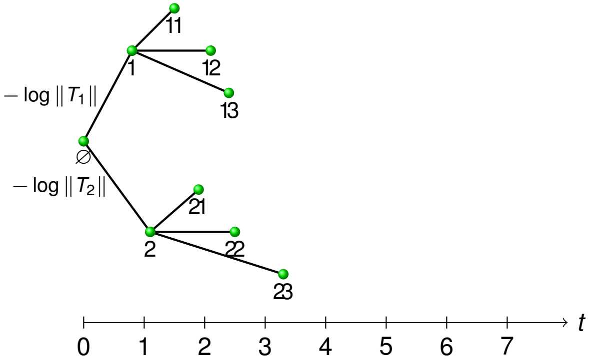

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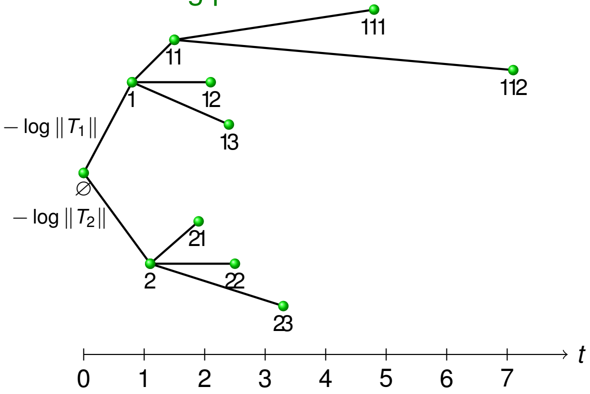

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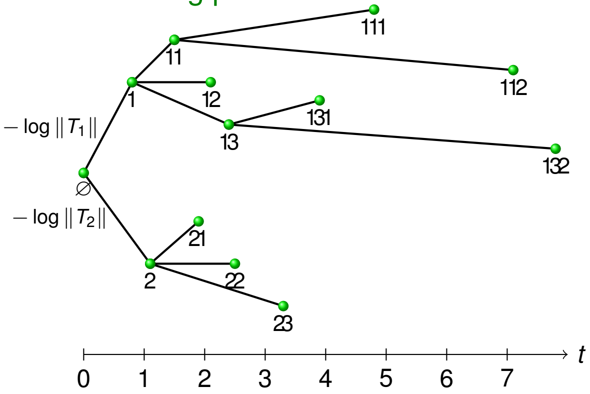

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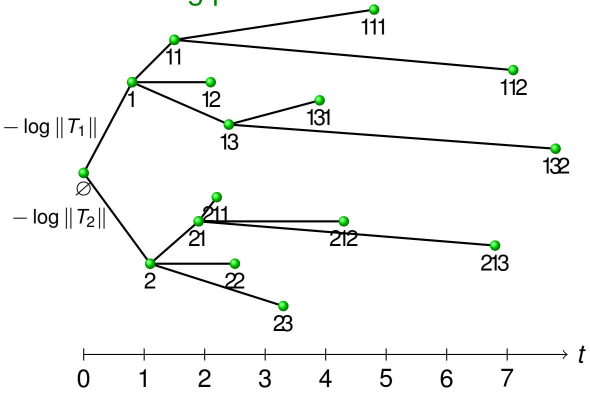

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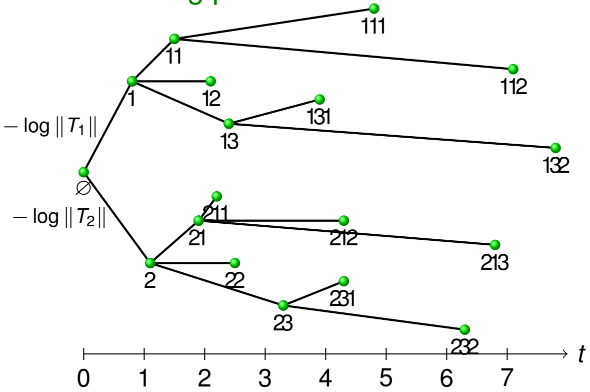

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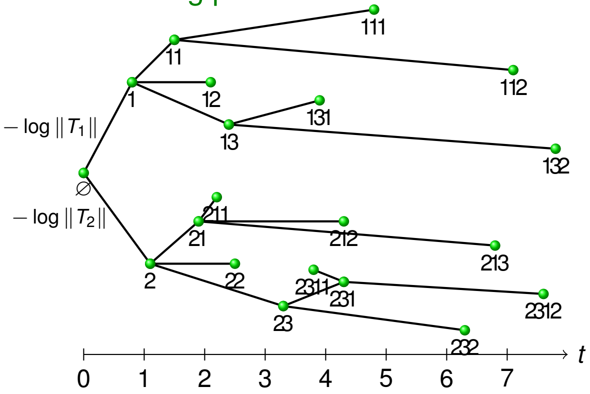

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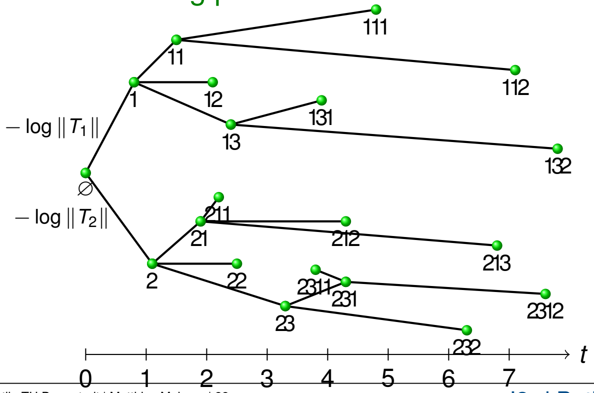

Last slide

The conjecture is proved

- when $\alpha \neq 1$;

Last slide

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Literature

G. Alsmeyer and M.
Fixed points of the smoothing transform: Two-sided solutions.
Probab. Theory Relat. Fields, 155:165–199, 2013.

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Thank you for your attention!