The minimum of a branching random walk outside the Cramer zone

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joint work with Julien Barral and Yueyun Hu

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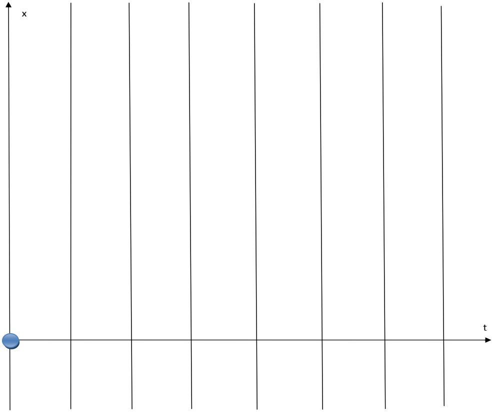
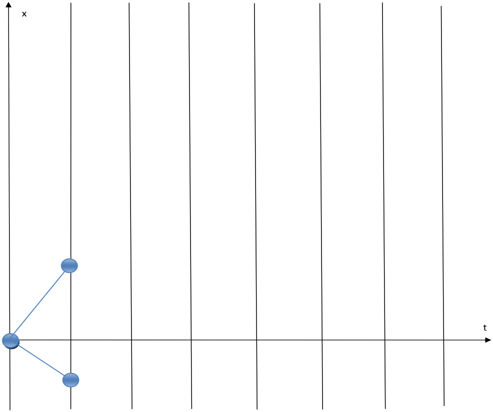
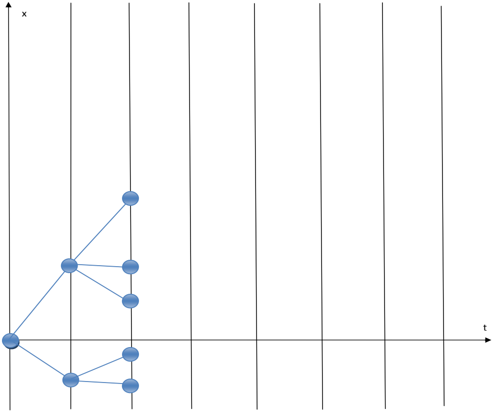
Generation 0



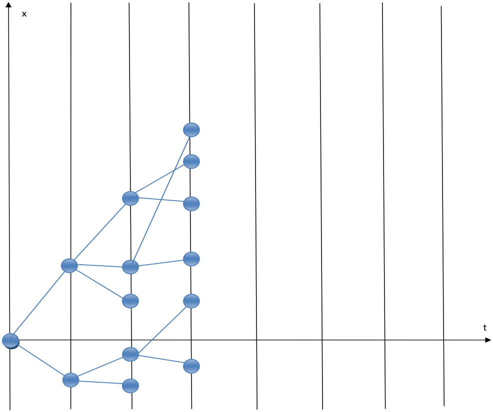
Generation 1



Generation 2



Generation 3



- Where is $M_n := \min_{|z|=n} V(z)$, the minimum of the branching random walk at time n?

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- The critical BRW:

$$\mathbf{E}\Big(\sum_{|z|=1} \mathrm{e}^{-V(z)}\Big) = 1, \qquad \mathbf{E}\Big(\sum_{|z|=1} V(z)\mathrm{e}^{-V(z)}\Big) = 0 \tag{1.1}$$

Theorem (Aïdékon, 2011)

There exists a constant $C_* > 0$ such that for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbf{P}(M_n \geq \frac{3}{2}\log n + x) = \mathbf{E}(\exp(-C_* \mathrm{e}^x D_\infty)), \tag{1.2}$$

where $D_\infty := \lim_{n\to\infty} \sum_{|z|=n} V(z)\mathrm{e}^{-V(z)}$ is the limit of the derivative martingale.

What is the Cramer zone?

- Let Φ be the log-generating function of the BRW

$$\Phi(t) := \log \mathbf{E}\Big(\sum_{|z|=1} \mathrm{e}^{-tV(z)} \Big), \quad t \geq 0. \tag{1.3}$$

Remark: Critical BRW means $\Phi(1) = 0$ and $\Phi'(1) = 0$.

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- What happens outside the Cramer zone?

A BRW under a first order phase transition

Let us define

$$\mathbf{E}(f[S_1]) := \mathbf{E}\Big(\sum_{|z|=1} f[V(z)]\mathrm{e}^{-V(z)}\Big) \quad \text{and} \quad \mathtt{m} := \mathbf{E}(S_1) = -\Phi'(1-) > 0.$$

- Let $\alpha > 1$. We assume that

$$\mathbf{P}(S_1 \le x) = \int_{-\infty}^{x} |y|^{-\alpha-1} l(y) dy, \quad \forall x \le x_0 < 0, \qquad (1.4)$$

  with l is a slowly varying function.

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  with l is a slowly varying function.
- Exemple: Let X, a random variable satisfying (1.4). Let V_1, V_2 two independent random variables with the law of $\frac{1}{2}\mathrm{e}^{x}\mathbf{P}_X(dx)$. Then the BRW built with $\mathcal{L} := \sum_{i\in\{1,2\}} \delta_{\{V_i\}}$ satisfy (1.4).

The main result

- For any $n \in \mathbb{N}$ let $\alpha_n := (\alpha + 1)\log n - \log l(n)$.

Theorem (Barral, Hu, M.)

Assume (1.4) and (???). For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{P}(M_n \geq \alpha_n + x) = \mathbf{E}(\exp(-c_* \mathrm{e}^x W_\infty)), \tag{2.1}$$

with $W_\infty := \lim_{n\to\infty} \sum_{|z|=n} \mathrm{e}^{-V(z)} > 0$, the limit of the additive martingale at the critical point.

- To establish (2.1) it suffices to know the tail distribution of the minimum M_n.

Proposition (Barral, Hu, M.)

Assume (1.4) and (???). There exists $c_* > 0$ *such that:*

$$\lim_{x\to\infty}\lim_{n\to\infty} \mathrm{e}^x \mathbf{P}\big(M_n \le \alpha_n - x\big) = c_*. \tag{3.1}$$

- Remark: for a critical BRW, Aïdékon proved that

$$\lim_{x\to\infty}\lim_{n\to\infty} \frac{\mathrm{e}^x}{x} \mathbf{P}\big(M_n \le \frac{3}{2}\log n - x\big) = C_*.$$

- It explains why the critical additive martingale appears in (2.1) instead of the derivative martingale.

An upper bound for the tail of distribution of M_n

Lemma (Barral, Hu, M.)

Assume (1.4). There exists some positive consant K *such that for all* $n \geq 2$ *and* $x \geq 0$,

$$\mathbf{P}\big(M_n \leq \alpha_n - x\big) \leq K\mathrm{e}^{-x}. \tag{3.2}$$

- We have not assumed the condition (???).
- We will sharpen the inequality

$$\begin{aligned}\mathbf{P}\big(M_n \leq \alpha_n - x\big) &\leq \mathbf{E}\Big(\sum_{|z|=n} 1_{\{V(z)\leq \alpha_n - x\}}\Big) \\ &= \mathbf{E}\big(\mathrm{e}^{S_n} 1_{\{S_n \leq \alpha_n - x\}}\big) \leq \mathrm{e}^{-x}\mathrm{e}^{\alpha_n}\mathbf{P}\big(S_n \leq \alpha_n - x\big).\end{aligned}$$

Lemma

Assume (1.4). There exists $K > 0$ such that for all n large enough

$$\mathbf{P}(0 \leq S_n \leq 1, \min_{j \leq n} S_j \geq 0) \leq K \mathrm{e}^{-\alpha_n} \tag{3.3}$$

- With an overwhelming probability the random walk $(S_n)_{n \geq 0}$ achieves a big negative jump of size $-\mathtt{m}n$.

The assumption (???)

- A renewal theorem is necessary to obtain the exact tail distribution.
- Let $\mathbf{Q}$ the probability measure defined by $\mathbf{Q}(w_n = u|\mathcal{F}_n) = \frac{\mathrm{e}^{-V(u)}}{W_n}$.
- For any $f : \mathbb{R} \to \mathbb{R}_+$ measurable with compact support

$$\lim_{z \to -\infty} \mathbf{E_Q}[\mathrm{e}^{-\sum_{|v|=1, v \neq w_1} f(V(w_1) - V(v))} \Big| V(w_1) = z]$$
$$= \int \Xi(d\theta) \mathrm{e}^{-\int_{\mathbb{R}} f(x)\theta(x)dx}.$$

where Ξ is the distribution of some point process on $\mathbb{R} \cap \{-\infty\}$.