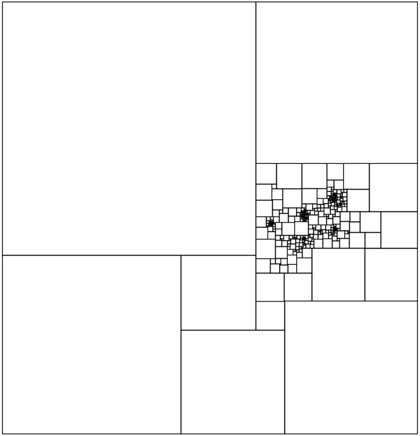
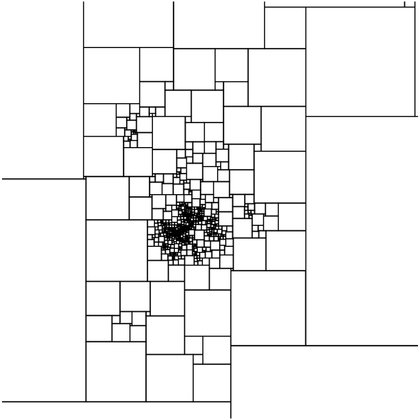
Random infinite squarings of rectangles

Louigi Addario-Berry Nicholas Leavitt





Bath/Paris III, June , 2014

Motivation

Random curves

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"Unif. random discrete curve":

Symmetric simple random walk.

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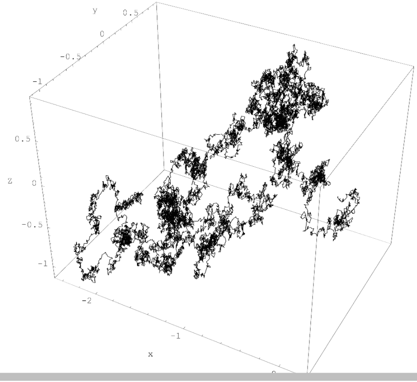
"Unif. random discrete curve": Symmetric simple random walk.

"Unif. random continuous curve": Brownian Motion



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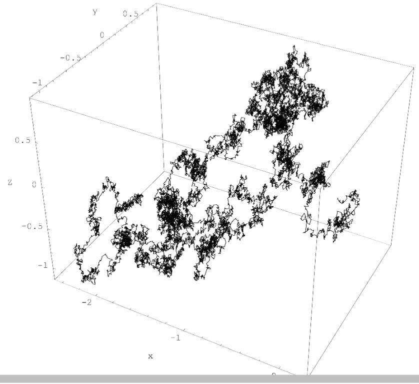

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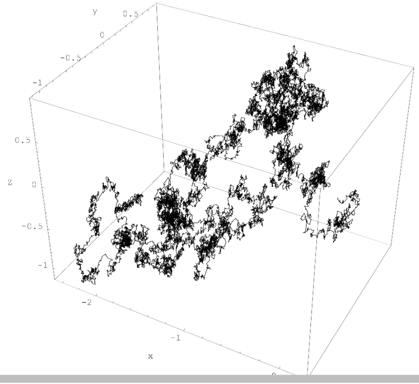

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Unif. random discrete surface: Random triangulation (map)

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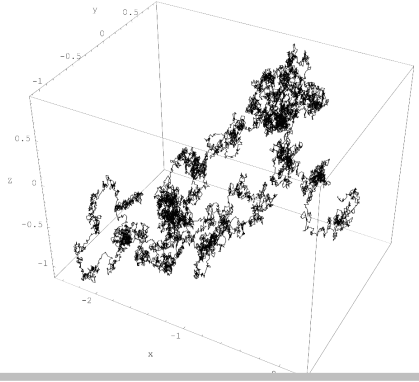

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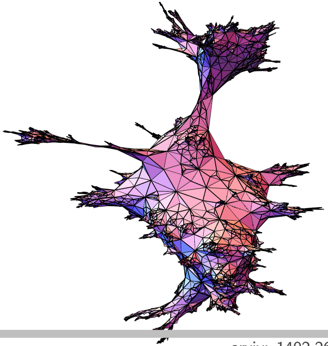

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Brownian motion: a *function* $B : [0, 1] \to \mathbb{R}$.

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Brownian Map

Brownian Map: an *equivalence class* of functions

$d : \mathbb{S}^2 \times \mathbb{S}^2 \to [0, \infty)$

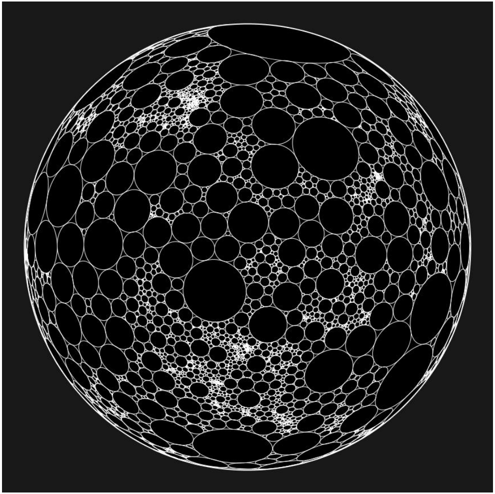
$d(a, b) =$distance from a to b.

$d \equiv d'$ if d and d' differ by a homeomorphism.

Canonical embeddings

We would like to have a "canonical" construction of a random metric Δ *on the sphere* $\mathbb{S}^2$ *in such a way that [the Brownian map has the law of]* $(\mathbb{S}^2, \Delta)$. *Furthermore, we expect* Δ *to behave well under the conformal transformations of the sphere." – Jean-Francois Le Gall, Proc. ICM 2014.*

Circle packings



Vertices: circles; edges: touching pairs.

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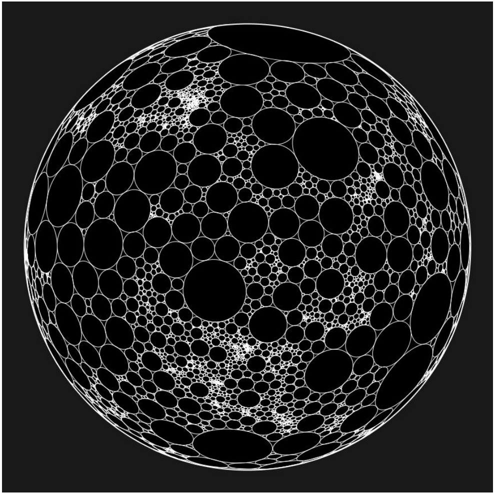

Vertices: circles; edges: touching pairs.
Embedding canonical up to conformal automorphisms.

J.-F. Le Gall, Proc. ICM 2014

"Suppose that, for ... $n \geq 2$, we have constructed a circle-packing embedding $\mathcal{C}_n$ of a uniformly distributed simple triangulation with n faces. Write $V(\mathcal{C}_n)$ for the vertex set of $\mathcal{C}_n$ and d^n_{gr} for graph distance on $V(\mathcal{C}_n)$....

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Conjecture. *One can construct the circle packing embeddings $\mathcal{C}_n$ in such a way that*

$$\sup_{x \in \mathbb{S}^2} \left(\min_{y \in V(\mathcal{C}_n)} |x - y| \right) \underset{n \to \infty}{\longrightarrow} 0$$

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in probability, and there exists a continuous random process $(\Delta(x,y))_{x,y \in \mathbb{S}^2}$ … such that

$$\sup_{x,y \in V(\mathcal{C}_n)} \left| \Delta(x,y) - (\frac{3}{2})^{1/4} n^{-1/4}\, d_{\mathrm{gr}}^n(x,y) \right| \underset{n \to \infty}{\longrightarrow} 0$$

in probability.

Canonical embeddings via *squarings*

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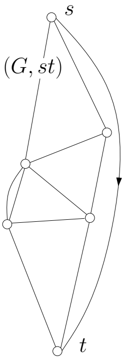

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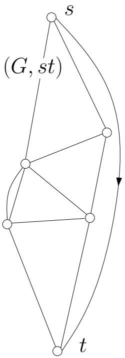

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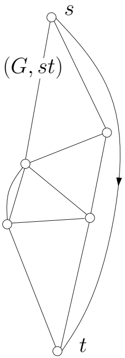

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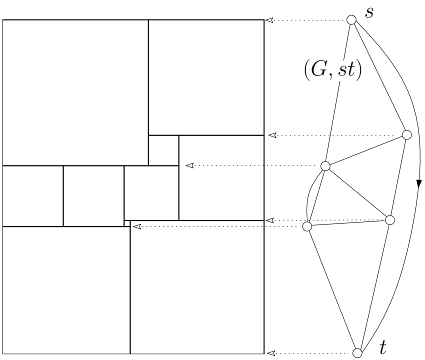

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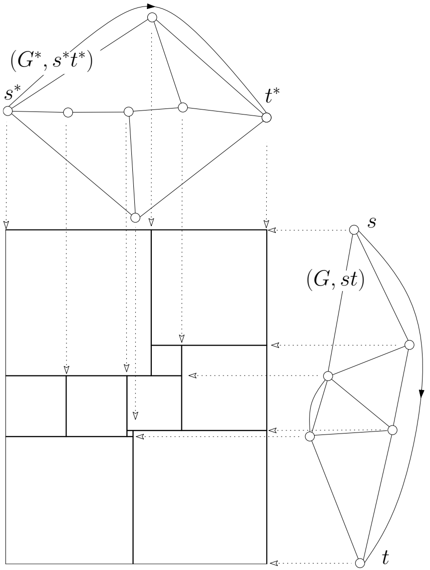
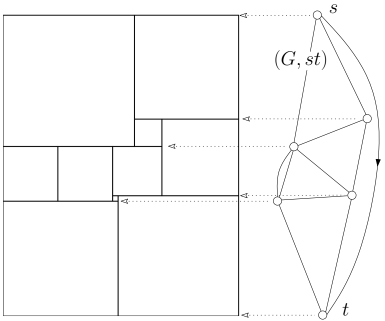
4. Square borders at $y = p(u), y = p(v)$ and $x = p(u^*), y = p(v^*)$.



Squaring examples



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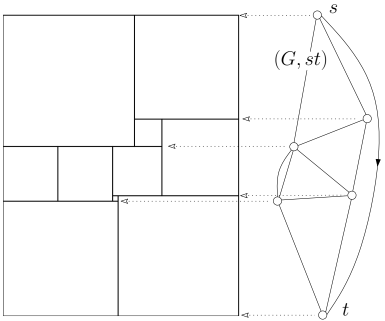

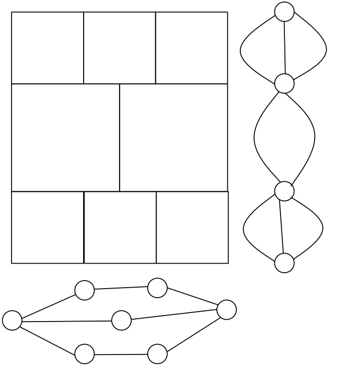

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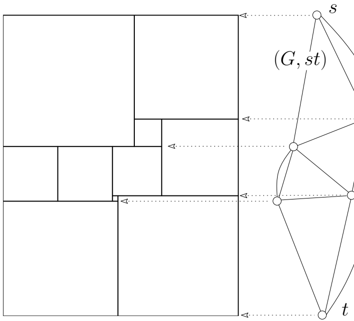

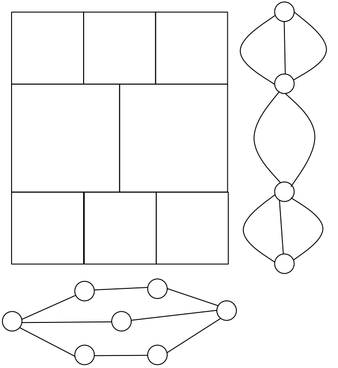

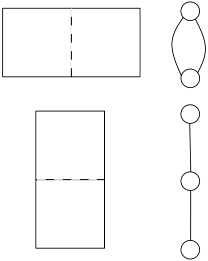

Our results

Theorem

Can "explicitly" construct a stochastic process $(S_n, 1 \le n \le \infty) = (S(\mathrm{G}_n), 1 \le n \le \infty)$ *with* $\mathrm{G}_n = (G_n, st)$ *such that*

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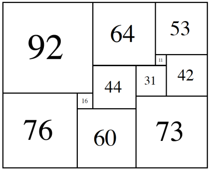
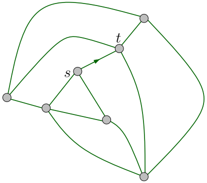
4. *$S_n \to S_\infty$ almost surely, for the Hausdorff distance.***
5. *S_∞ a.s. has exactly one point of accumulation.****

Proof idea

- To grow squarings, grow maps (BSST '40).





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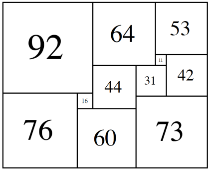

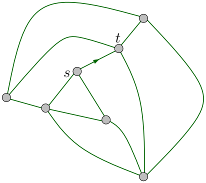

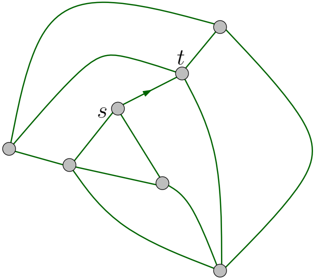

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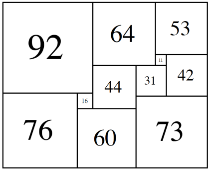

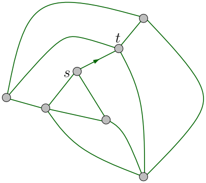

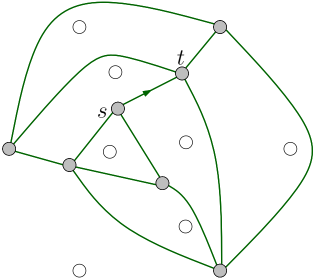

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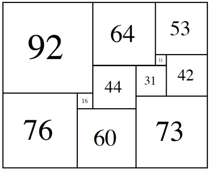

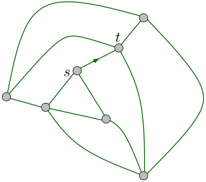

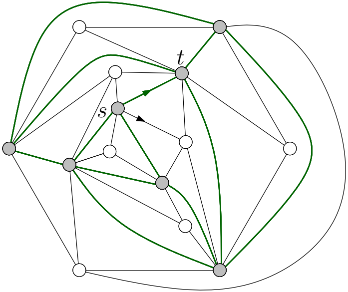

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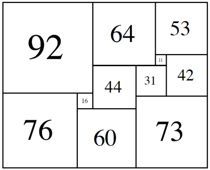

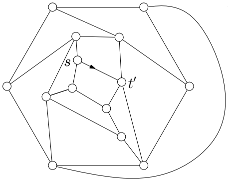

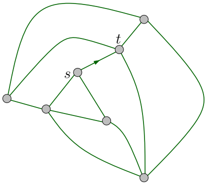

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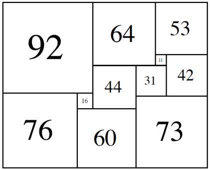

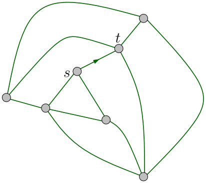

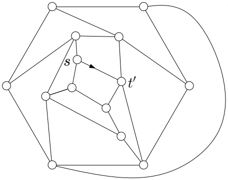

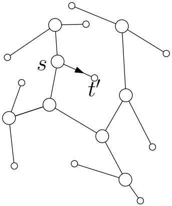

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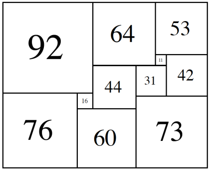

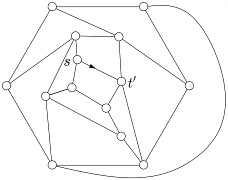

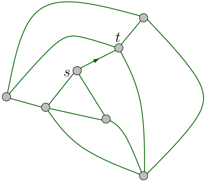

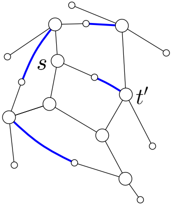

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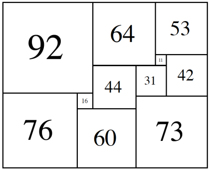

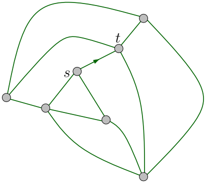

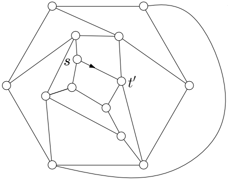

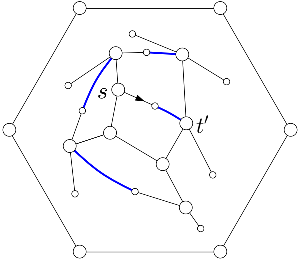

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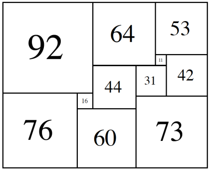

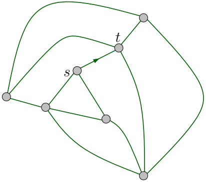

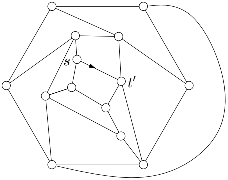

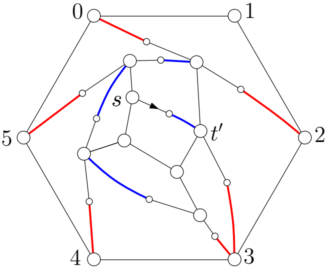

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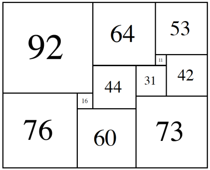

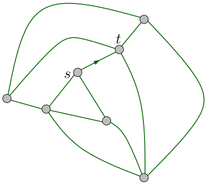

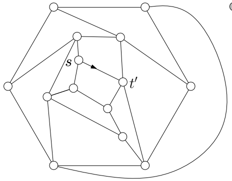

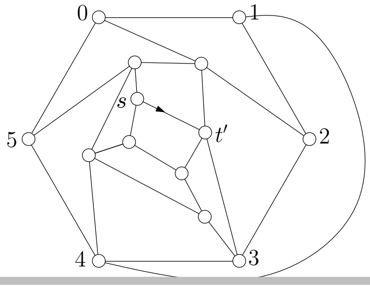

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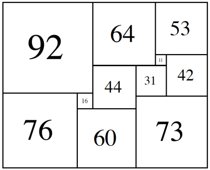

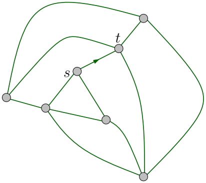

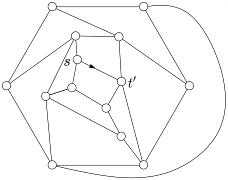

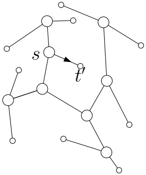

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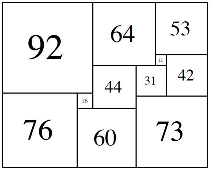

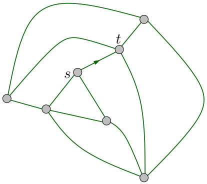

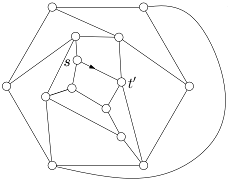

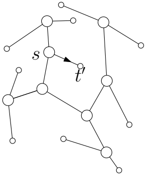

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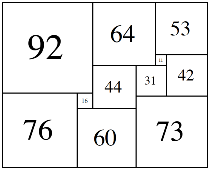

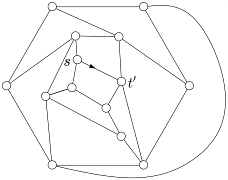

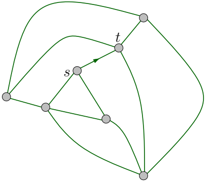

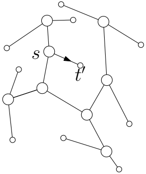

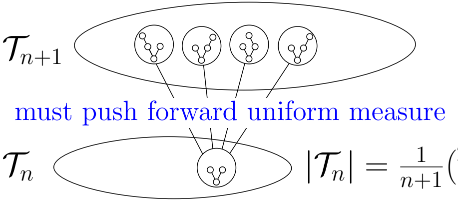

$$|\mathcal{T}_n| = \frac{1}{n+1}\binom{2n}{n}$$

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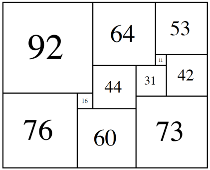
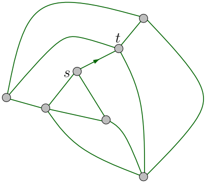
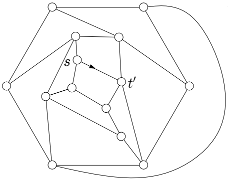
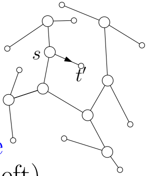
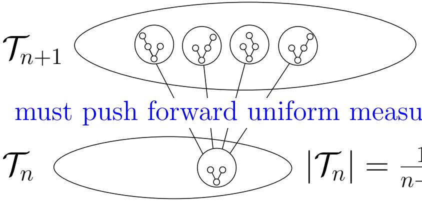








Suffices to calculate
$\mathbf{P}_n(\text{Grow left} \mid k \text{ on left})$



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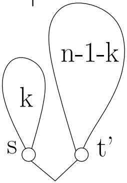

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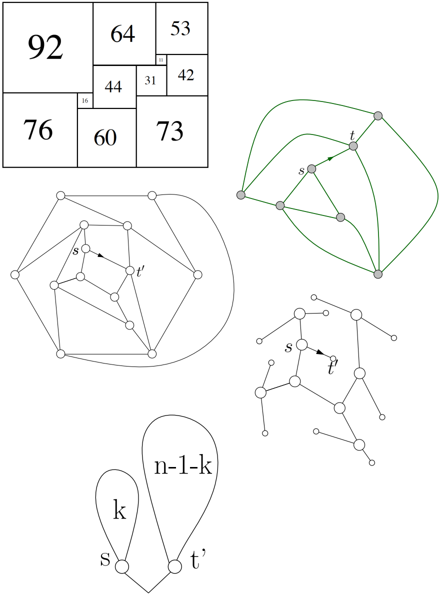

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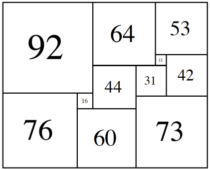
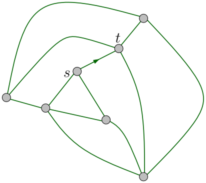
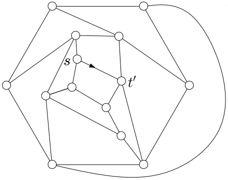
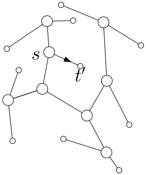
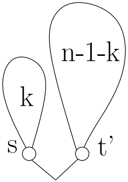




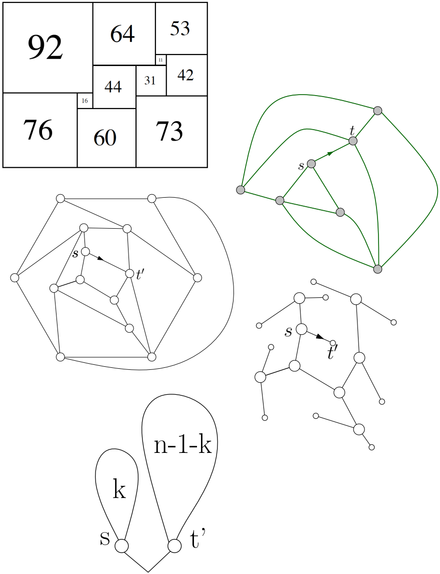






So growing binary trees leads to a growth rule for squarings.

Proof idea Growing binary trees yields growth rule for squarings



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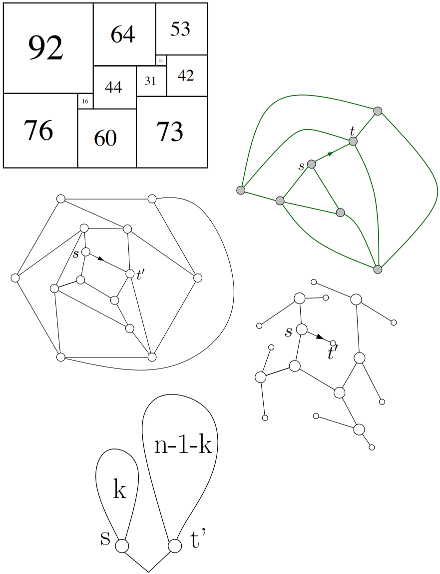

Then prove:

- Growth rule is *local*: only affects squares incident to a single line segment.

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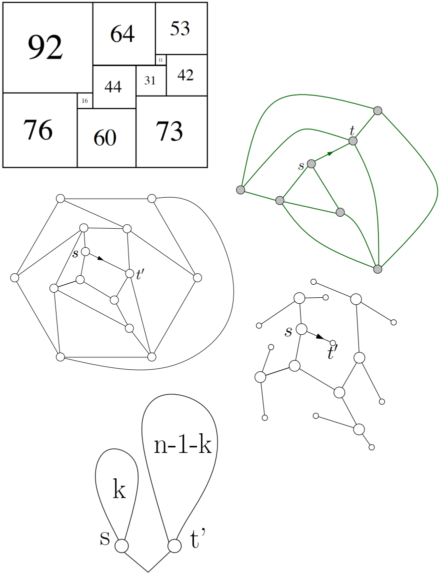

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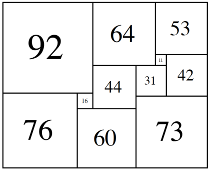

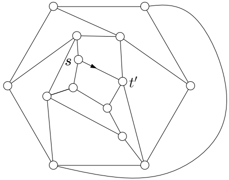

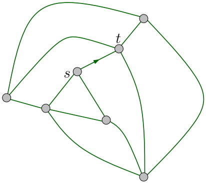

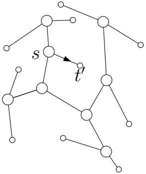

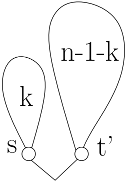

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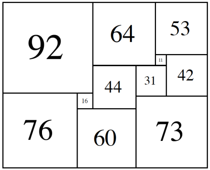

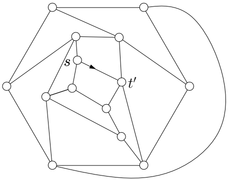

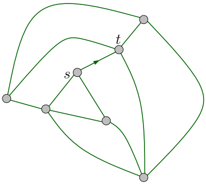

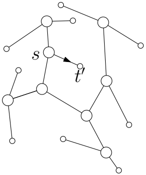

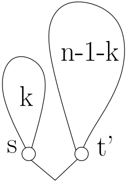

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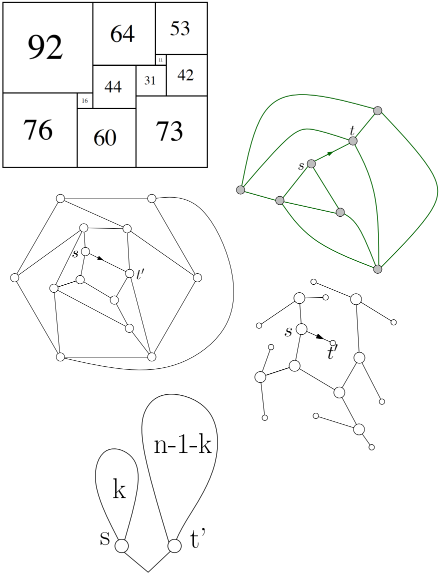

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- He and Schramm (1995): if the squaring is "parabolic" then it has only one accum. point.
- Parabolicity can be proved using recurrence of G_∞.

Open questions

⋆ There is a unique translation and scaling under which the image S_n' of S_n is centred at 0 and such that when S_n' is stereographically projected to the Riemann sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, the image of the unbounded region of $\mathbb{R}^2 \setminus S_n$ has area $1/n$. Apply this transformation, and let μ_n be the measure on $\mathbb{C}^*$ obtained by letting each connected component of $\mathbb{C}^* \setminus S_n'$ have measure $1/n$.[1] Then μ_n should converge weakly to a measure μ on $\mathbb{C}^*$ which is some version of the Liouville quantum gravity measure (possibly the "γ-unit area quantum sphere measure with $\gamma = \sqrt{8/3}$", introduced by Sheffield (2010). In particular, μ should satisfy a version of the KPZ dimensional scaling relation.

⋆ We expect that the box-counting dimension of S_∞ is a.s. well-defined and constant. More precisely, write n_ϵ for the number of balls of radius ϵ required to cover S_∞. We expect that $\log n_\epsilon / \log(1/\epsilon) \to c$ almost surely, where c is non-random. Is this true? If so, what is c? Is $c > 1$? (Note that for the Hausdorff dimension, if $(C_n, n \in \mathbb{N})$ are measurable sets in $\mathbb{R}^2$ then $\dim_{\mathrm{Haus}}(\bigcup_{n\in\mathbb{N}} C_n) = \sup_{n\in\mathbb{N}} \dim_{\mathrm{Haus}}(C_n)$. Since S_∞ is a countable union of line segments, it follows that $\dim_{\mathrm{Haus}}(S_\infty) = 1$ almost surely.)

⋆ Let Z be the a.s. unique accumulation point of S_∞. Can the law of Z be explicitly described?

⋆ Write $G_\infty(\epsilon)$ for the graph induced by those vertices for which all incident squares are disjoint from $B(Z, \epsilon)$. How quickly does $G_\infty(\epsilon)$ grow as ϵ decreases? Relatedly, how does the diameter of $G_\infty(\epsilon)$ grow? Existing results about random maps suggest that if the diameter grows as $\epsilon^{-\alpha}$ then the volume should grow as $\epsilon^{-4\alpha}$.

⋆ The structure of S_∞ near Z should be independent of its structure near the root; here is one question along these lines. Reroot G_∞ by taking one step along a random walk path from the root, write $\hat{S}_\infty$ for the resulting squaring and $\hat{Z}$ for its point of accumulation. Then recenter S_∞ and $\hat{S}_\infty$ so that Z and $\hat{Z}$ sit at the origin. Does $\epsilon^{-1} d_H(S_\infty \cap B(0,\epsilon), \hat{S}_\infty \cap B(0,\epsilon)) \to 0$ almost surely, as $\epsilon \to 0$? Here d_H denotes Hausdorff distance.

⋆ Let $e_n(1), \ldots, e_n(k)$ be independent, uniformly random oriented edges of the contacts graph $R(S_n)$, and for $1 \le i \le k$ let $r_n(i)$ be the ratio of the side length of the "tail square" of $e_n(i)$ to that of its "head square". The vector $(r_n(i), 1 \le i \le k)$ should converge in distribution to a limit $(r(i), 1 \le i \le k)$, whose entries are iid. This would be a very small first step towards establishing that the random squaring in some sense "looks like the exponential of a Gaussian free field".

⋆ Let A_n be the adjacency matrix of G_n. The areas of squares may be calculated as determinants of minors of A_n. However, these determinants grow very quickly, and even finding logarithmic asymptotics seems challenging. A simpler, still challenging project is to study the determinant of any principal minor of A_n or, equivalently, to study the number of spanning trees of G_n.

⋆ The height of S_∞ is 1 but its width W_∞ is random, and by considering the graph structure near the root of G_∞ it is not hard to see that W_∞ is an honest random variable (rather than a.s. constant) On the other hand, duality implies that W_∞ and $1/W_\infty$ have the same law. Can anything explicit be said about this law? In particular, is $\mathbb{P}(W_\infty = 1) > 0$?

⋆ Simulations suggest that for n large, S_n is unlikely to contain four squares with common intersection. Does this probability indeed tend to zero as n becomes large? This question looks innocent. However, recall that such intersections are the reason the function sending a rooted planar graph to its squaring is non-invertible. A positive answer would constitute substantial progress towards proving an asymptotic formula, conjectured by Tutte (1963), for the number of perfect squarings with n squares.

⋆ Let $\hat{S}_n$ be uniformly distributed over squarings of a rectangle with n squares. Does $\hat{S}_n$ converge in distribution to S_∞ for the Hausdorff distance? This follows if the laws of S_n and $\hat{S}_n$ are close, which would itself follow from a positive answer to the previous question.

⋆ The behaviour of the simple random walk on G_∞ is also of interest. How do quantities such as $\mathbb{P}(X_t = X_0)$, $d_{G_\infty}(X_0, X_t)$, and $\#\{X_s, 0 \le s \le t\}$ scale in t?

⋆ It seems likely that $R(S_\infty)$ is recurrent: is it? Here is one tempting argument for recurrence; its incorrectness was pointed out to us by

Thank you

Figure : A partial cubing of a cube.