The mass of super-Brownian motion upon exiting balls and Sheu's compact support condition

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Super-Brownian motion

Consider a finite-measure-valued strong Markov process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ whose evolution is characterised via its log-laplace semi-group: For all $f \in C_b^+(\mathbb{R})$, the space of positive, uniformly bounded, continuous functions on $\mathbb{R}^d$, and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ (the space of finite measure on $\mathbb{R}^d$),

$$-\log \mathbf{E}_\mu(\mathrm{e}^{-\langle f, X_t \rangle}) = \int_{\mathbb{R}} v_f(x,t)\mu(\mathrm{d}x), \ \ t \geq 0,$$

where $v_f(x,t)$ is the unique positive solution to the evolution equation for $x \in \mathbb{R}$ and $t > 0$

$$\frac{\partial}{\partial t} v_f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v_f(x,t) - \psi(v_f(x,t)),$$

with initial condition $v_f(x,0) = f(x)$. The branching mechanism ψ satisifes:

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x)\nu(\mathrm{d}x), \tag{1}$$

for $\lambda \geq 0$ where $\alpha = -\psi'(0^+) \in (0,\infty)$, $\beta \geq 0$ and ν is a measure concentrated on $(0,\infty)$ which satisfies $\int_{(0,\infty)} (x \wedge x^2)\nu(\mathrm{d}x) < \infty$.

Super-Brownian motion

- Another way of representing the log-Laplace semi-group evolution is via the integral equation:

$$v_f(x,t) = \mathbb{E}_x[f(\xi_t)] - \mathbb{E}_x\Big[\int_0^t \psi(v_f(\xi_z, t-z))\,\mathrm{d}z\Big],$$

and $((\xi_z, z \geq 0), \mathbb{P}_x)$ is an $\mathbb{R}^d$-Brownian motion with $\xi_0 = x$

- Choosing $f = 1$ produces the log-Laplace exponent a CSBP with branching mechanism ψ. That is to say the total mass process, $||X_t|| := \langle 1, X_t \rangle$, $t \geq 0$, is a CSBP.
- This super-BM is the continuum analogue of Branching Brownian motion with a general off-spring distribution (including allowing for no offspring w.p.p.).
- The constant $-\psi'(0+) = \alpha$ gives us the growth and hence process is (sub/super)-critical. Largely indifferent to criticality in this talk.

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Mass decay and Grey's condition

- Following the same property of all (non-monotone) CSBPs:

$$\mathbf{P}_\mu(\lim_{t\to\infty}||X_t|| = 0 \mid ||X_0|| = x) = e^{-\lambda^*||\mu||},$$

where $\psi(\lambda^*) = 0$ and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$.

- On the event $\{||X_t|| \to 0\}$, Grey (1974) gives us a nice dichotomy between the two ways in which this can happen: either

$$\{\exists T(\omega) > 0 \text{ s.t. } ||X_{T+t}|| = 0 \ \forall t \geq 0\} \text{ (extinction)}$$

$$\text{or } \{||X_t|| \to 0 \text{ and } ||X_t|| > 0 \ \forall t > 0\} \text{ (extinguishing)}$$

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Sheu's condition

- For the case of super-Brownian motion, Sheu (1994) offers an additional unusual condition for the event of compact support: Let

$$\mathcal{S} = \bigcup_{t\geq 0} \text{supp} X_t$$

Then for all compactly supported $\mu \in \mathcal{M}_F(\mathbb{R}^d)$,

$$\mathbf{P}_\mu(\mathcal{S} \text{ is compact}) = e^{-\lambda_* ||\mu||}$$

if and only if

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^{\lambda} \psi(\theta)\, d\theta}} \, d\lambda < \infty$$

and otherwise $\mathbb{P}_\mu(\mathcal{S} \text{ is compact}) = 0$.

- What is the relation between this condition and Grey's condition? Sheu's condition comes out of PDE analysis and it is unclear.
- What is the relation between $\{\mathcal{S} \text{ is compact}\}$ and $\{||X_t|| \to 0\}$?

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First passage branching process

- Fix an initial radius $r > 0$ and let $D_s := \{x \in \mathbb{R}^d : ||x|| < s\}$ be the open ball of radius $s \geq r$ around the origin.
- According to Dynkin's theory of exit measures we can describe the mass of X as it first exits the growing sequence of balls $(D_s, s \geq r)$ as a sequence of random measures on $\mathbb{R}^d$, known as branching Markov exit measures.
- We denote this sequence of branching Markov exit measures by $\{X_{D_s}, s \geq r\}$. Informally, the measure X_{D_s} is supported on the boundary ∂D_s and it is obtained by 'freezing' mass of the super-Brownian motion when it first hits ∂D_s. If X were a branching Brownian motion, then X_{D_s} would be a stopping line *à la* Chauvin-Neveu.
- For $s \geq r$, let $Z_s := ||X_{D_s}||$ denote the mass that is 'frozen' when it first hits the boundary of the ball D_s. We can then define the mass process $(Z_s, s \geq r)$ which uses the radius s as its time-parameter.

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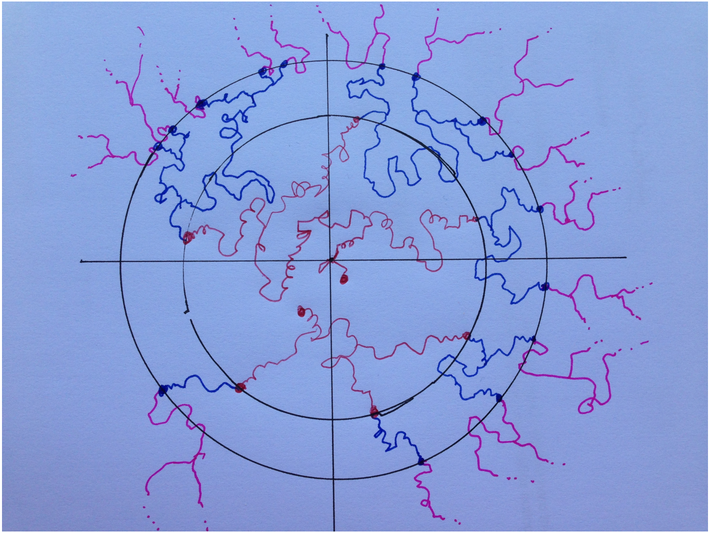
Z is a time-inhomogenous CSBP



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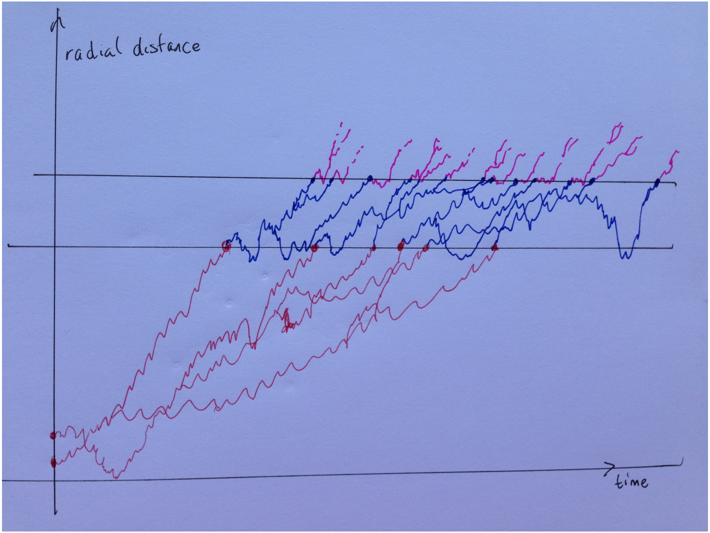

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Theorem

Let $r > 0$. The process $Z = (Z_s, s \geq r)$ is a time-inhomogeneous continuous-state branching process. Let $r > 0$ and $\mu \in \mathcal{M}_F(\partial D_r)$ with $||\mu|| = a$. Then, for $s \geq r$, we have

$$E_{a,r}[\mathrm{e}^{-\theta Z_s}] = \mathrm{e}^{-u(r,s,\theta)a}, \quad \theta \geq 0,$$

where the Laplace functional $u(r,s,\theta)$ satisfies

$$u(r,s,\theta) = \theta - \int_r^s \Psi(z, u(z,s,\theta))\, \mathrm{d}z,$$

for a family of branching mechanisms $(\Psi(r,\cdot), r > 0)$ satisfying the PDE

$$\begin{aligned} \frac{\partial}{\partial r}\Psi(r,\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) &= 2\psi(\theta) \\ \Psi(r,\lambda^*) &= 0, \end{aligned}$$

for $r > 0$, $\theta \in (0,\infty)$.

Asymptotic behaviour of Z

Proposition

(i) For (sub)critical ψ, we have, for $0 < r \leq s$,

$$\Psi(r, \theta) \leq \Psi(s, \theta) \quad \textit{for all} \ \ \theta \geq 0.$$

(ii) For supercritical ψ, we have, for $0 < r \leq s$,

$$\Psi(r, \theta) \geq \Psi(s, \theta) \quad \textit{for all} \ \ \theta \leq \lambda^*$$
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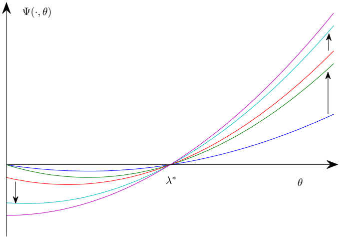

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Lemma

For each $\theta \geq 0$, the limit $\lim_{r\uparrow\infty} \Psi(r,\theta) = \Psi_\infty(\theta)$ is finite and the convergence holds uniformly in θ on any bounded, closed subset of $\mathbb{R}_+$.
For any $\theta \geq 0$, we have

$$\Psi_\infty(\theta) = 2\,\mathrm{sgn}(\psi(\theta)) \sqrt{\int_{\lambda^*}^{\theta} \psi(\lambda)\, \mathrm{d}\lambda},$$

with $\lambda^* = 0$ in the (sub)critical case.

$$\begin{aligned} \frac{\partial}{\partial r}\Psi(r,\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) &= 2\psi(\theta) \\ \Psi(r,\lambda^*) &= 0, \end{aligned}$$

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Lemma

Denote by $((Z_s^\infty, s \geq 0), P^\infty)$ the standard CSBP associated with the limiting branching mechanism Ψ_∞, with unit initial mass at time 0.
Then, for any $s > 0$, $\theta \geq 0$,

$$\lim_{r\to\infty} E_{r,1}[\mathrm{e}^{-\theta Z_{r+s}}] = E^\infty[\mathrm{e}^{-\theta Z_s^\infty}].$$

Sheu's condition is Grey's condition

- Sheu's condition is Grey's condition for Z^∞.

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^\lambda \psi(\theta)\ \mathrm{d}\theta}}\ \mathrm{d}\lambda = \int^\infty \frac{1}{\Psi^\infty(\lambda)}\mathrm{d}\lambda.$$

- There is no hierarchy between $\{||X_t|| \to 0\}$ and $\{\mathcal{S}$ is compact$\}$.
- Take e.g. the supercritical branching mechanism $\psi(\lambda) = \lambda - (\lambda+2)^\alpha + 2^\alpha$ for $\alpha \in (0,1)$. This branching mechanism respects $\int^\infty 1/\psi(\lambda)\mathrm{d}\lambda = \infty$ (extinguishing) but $\int^\infty 1/(\int_{\lambda^*}^\lambda \psi(\theta)\ \mathrm{d}\theta)^{1/2}\mathrm{d}\lambda = \infty$ (no compact support).
- In principle it could happen that ψ is such that we have a process that becomes extinct but which is not compactly supported.

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Martingales

- On the one hand, using the semi-group equations,

$$M_s^\lambda = e^{-\lambda^* Z_s} - \int_r^s \Psi(v,\lambda^*) Z_v e^{-\lambda^* Z_v} \mathbf{1}_{\{Z_v<\infty\}} \mathrm{d}v, \quad s \geq r,$$

is a martingale.

- On the other hand $\exp\{-\lambda^* ||X_t||\}$, $t \geq 0$ is a martingale since

$$\mathbf{E}_\mu[\mathbf{1}_{\{||X_u||\to 0\}} \,|\sigma(||X_s||, s\leq t) = \mathrm{e}^{-\lambda^*||X_t||}, \qquad t\geq 0,$$

and hence so is

$$\mathbf{E}_\mu[\mathbf{1}_{\{||X_u||\to 0\}} \mid \sigma(||X_{D_v}||, r\leq v\leq s)] = e^{-\lambda^*||X_{D_s}||} = e^{-\lambda^* Z_s},$$

- Comparing tells us that $\Psi(r,\lambda^*) = 0$ for all $r > 0$.

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Branching mechanism PDE

- On the one hand: Let $r > 0$ and $\mu \in \mathcal{M}_F(\partial D_r)$ with $||\mu|| = a$. Then, for $s \geq r$, we have

$$E_{a,r}[e^{-\theta Z_s}] = e^{-u(r,s,\theta)a}, \quad \theta \geq 0,$$

where the Laplace functional $u(r,s,\theta)$ satisfies

$$u(r,s,\theta) = \theta - \int_r^s \Psi(z, u(z,s,\theta))\, \mathrm{d}z,$$

- On the other hand: Recalling that the radial part of an $\mathbb{R}^d$-Brownian motion is a Bessel process, Dynkin's semigroup theory for branching Markov exit measures gives us

$$u(r,s,\theta) = \theta - \mathbb{E}_r^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_\ell, s, \theta))\, \mathrm{d}\ell, \; 0 < r \leq s, \; \theta \geq 0,$$

where $(R, \mathbb{P}^{\mathrm{R}})$ is a d-dimensional Bessel process and $\tau_s := \inf\{l > 0 : R_l > s\}$ its first passage time above level s

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Branching mechanism PDE

- Define $\varphi(s) = \int_0^{r^2 s} R_\ell^{-2} \mathrm{d}\ell$, $s \geq 0$, then

$$B_s = \log(r^{-1} R_{r^2 \varphi^{-1}(s)}), \; s \geq 0,$$

  is a one-dimensional Brownian motion with drift $\frac{d}{2} - 1$.

- the Last semi-group equation can be developed into

$$\begin{aligned} u(r,s,\theta) &= \theta - \mathbb{E}_{\log r} \int_0^{T_{\log s}} \psi(u(e^{B_l}, s, \theta)) e^{2B_\ell} \, \mathrm{d}\ell \\ &= \mathbb{E}_{\log r} \sum_{\log r \leq u \leq \log s} \int_0^{\zeta^{(u)}} \psi(u(e^{u - e_u(l)}, s, \theta)) e^{2(u - e_u(\ell))} \, \mathrm{d}\ell \\ &= \theta - 2 \int_r^s v^{1-d} \int_0^v \psi(u(z,s,\theta)) \, z^{d-1} \, \mathrm{d}z \, \mathrm{d}v. \end{aligned}$$

Branching mechanism PDE

- Define $\varphi(s) = \int_0^{r^2 s} R_\ell^{-2} \mathrm{d}\ell$, $s \geq 0$, then

$$B_s = \log(r^{-1} R_{r^2\varphi^{-1}(s)}), \;\; s \geq 0,$$

  is a one-dimensional Brownian motion with drift $\frac{d}{2} - 1$.

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Branching mechanism PDE

- Line up the two representations of the semi-group equation:

$$u(r,s,\theta) = \theta - 2\int_r^s v^{1-d} \int_0^v \psi(u(z,s,\theta))\; z^{d-1}\, \mathrm{d}z\, \mathrm{d}v.$$

and

$$u(r,s,\theta) = \theta - \int_r^s \Psi(z, u(z,s,\theta))\, \mathrm{d}z,$$

- Fiddling with derivatives in s, r and θ, gives the desired PDE

$$\begin{aligned} \frac{\partial}{\partial r}\Psi(r,\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) &= 2\psi(\theta) \\ \Psi(r,\lambda^*) &= 0, \end{aligned}$$

for $r > 0,\ \theta \in (0,\infty)$.

Branching mechanism PDE

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Thankyou


Thank
You
Merci
Gracias
Mèsi
Xièxie
Takk
aDank
BarakAllahuFiik
Grazie
Chokrane
Kiitos
Dziakuju
TerimaKasih
Rahmat
ThintKo
Buznyg
Blagodaram
TapadhLeat
Köszönöm
Murakoze
WaadMahadsantahay
Grazzi
Matondo
Mercé
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Tenki
Aabhar
Barkal
GratiasAgimus
Mauruuru
Aciū
Dankie
Tak
Sagolun
Motashakkeram
TänanVäga
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