Potential energy of biased random walks on trees

Yueyun Hu (Université Paris 13)
joint work with Zhan Shi (Paris 6)
Bath, June 2014

Plan of talk

1. From Random walk in random environment (RWRE) on $\mathbb{Z}$ to biased random walks (or RWRE) on trees ;
2. The asymptotic behaviors of a class of recurrent RWRE on trees
3. Potential energy and main result.
4. Proof.

1-d Random walks in random environments (Z_n)

Let $\omega = \{\omega_x, x \in \mathbb{Z}\}$ be a family of i.i.d. random variables (and no constant) taking values in $(0, 1)$. The ω plays the role of random environment.

Given ω, $\{Z_n, n \geq 0\}$ is a Markov chain taking values in $\mathbb{Z}$ starting from 0 with probability transition :

$$\mathbb{P}_\omega\Big(Z_{n+1} = y \Big| Z_n = x\Big) = \begin{cases} \omega_x, & \text{if } y = x + 1\,; \\ 1 - \omega_x, & \text{if } y = x - 1. \end{cases}$$

1-d Random walks in random environments (Z_n)

Let $\omega = \{\omega_x, x \in \mathbb{Z}\}$ be a family of i.i.d. random variables (and no constant) taking values in $(0,1)$. The ω plays the role of random environment.

Given ω, $\{Z_n, n \geq 0\}$ is a Markov chain taking values in $\mathbb{Z}$ starting from 0 with probability transition :

$$\mathbb{P}_\omega\Big(Z_{n+1} = y \Big| Z_n = x\Big) = \begin{cases} \omega_x, & \text{if } y = x+1\,; \\ 1 - \omega_x, & \text{if } y = x-1. \end{cases}$$

Asymptotic behaviors of (Z_n)

References

- P. Révész : *Random walk in random and non-random environments* (1st edition : 1990, 2nd edition : 2005)
- O. Zeitouni : Lecture notes Saint Flour 2001.
- A.S. Sznitman, M.Zerner ... (multi-dimensional case).

Recurrence/transience criteria : Solomon (1975)

- (Z_n) is recurrent if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) = 0$;
- $Z_n \to \infty$ if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) < 0$.

Asymptotic behaviors of (Z_n)

References

- P. Révész : *Random walk in random and non-random environments* (1st edition : 1990, 2nd edition : 2005)
- O. Zeitouni : Lecture notes Saint Flour 2001.
- A.S. Sznitman, M.Zerner ... (multi-dimensional case).

Recurrence/transience criteria : Solomon (1975)

- (Z_n) is recurrent if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) = 0$;
- $Z_n \to \infty$ if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) < 0$.

How big is (Z_n) ?

Transient case : Kesten, Kozlov and Spiter (1976)

when $Z_n \to \infty$, $Z_n \approx n^\rho$. The exponent ρ is explicitly determined by the law of ω_x and can vary in $(0, 1]$.

Recurrent case : Sinai (1982)'s localization

when (Z_n) is recurrent, $\frac{Z_n}{\log^2 n}$ converges in law (to some non-degenerated law, explicitly computed by Kesten (1986) and Golosov (1986)).

Question : What happens on trees ?

We may find both (sub)diffusive (n^ϱ with $0 < \varrho \leq \frac{1}{2}$) and slow movement behaviors in a class of recurrent RWREs on trees.

How big is (Z_n) ?

Transient case : Kesten, Kozlov and Spiter (1976)

when $Z_n \to \infty$, $Z_n \approx n^\rho$. The exponent ρ is explicitly determined by the law of ω_x and can vary in $(0, 1]$.

Recurrent case : Sinai (1982)'s localization

when (Z_n) is recurrent, $\frac{Z_n}{\log^2 n}$ converges in law (to some non-degenerated law, explicitly computed by Kesten (1986) and Golosov (1986)).

Question : What happens on trees ?

We may find both (sub)diffusive (n^ϱ with $0 < \varrho \leq \frac{1}{2}$) and slow movement behaviors in a class of recurrent RWREs on trees.

How big is (Z_n) ?

Transient case : Kesten, Kozlov and Spiter (1976)

when $Z_n \to \infty$, $Z_n \approx n^\rho$. The exponent ρ is explicitly determined by the law of ω_x and can vary in $(0, 1]$.

Recurrent case : Sinai (1982)'s localization

when (Z_n) is recurrent, $\frac{Z_n}{\log^2 n}$ converges in law (to some non-degenerated law, explicitly computed by Kesten (1986) and Golosov (1986)).

Question : What happens on trees ?

We may find both (sub)diffusive (n^ϱ with $0 < \varrho \leq \frac{1}{2}$) and slow movement behaviors in a class of recurrent RWREs on trees.

RWRE on trees

Random environments

Let $\mathbb{T}$ be a supercritical Galton-Watson tree rooted at $\varnothing$ and $\omega = \{(\omega(x,y), y \in \mathbb{T})_{x\in\mathbb{T}}\}$ be a family of random variables such that $\sum_{y\in T : y\sim x} \omega(x,y) = 1$, $\omega(x,y) > 0$ if $x \sim y$ ($x \sim y$ means x and y are adjacent).

Random walk in random environment (X_n) on a tree :

Conditioned on ω, (X_n) is a Markov chain taking values in $\mathbb{T}$ with probability transition :

$$\mathbb{P}_\omega\Big(X_{n+1} = y \,\Big|\, X_n = x\Big) = \omega(x,y).$$

RWRE on trees

Random environments

Let $\mathbb{T}$ be a supercritical Galton-Watson tree rooted at $\varnothing$ and $\omega = \{(\omega(x,y), y \in \mathbb{T})_{x\in\mathbb{T}}\}$ be a family of random variables such that $\sum_{y\in T: y\sim x} \omega(x,y) = 1$, $\omega(x,y) > 0$ if $x \sim y$ ($x \sim y$ means x and y are adjacent).

Random walk in random environment (X_n) on a tree :

Conditioned on ω, (X_n) is a Markov chain taking values in $\mathbb{T}$ with probability transition :

$$\mathbb{P}_\omega\Big(X_{n+1} = y \,\big|X_n = x\Big) = \omega(x,y).$$

Notations

For each vertex $x \in \mathbb{T} \backslash \{\varnothing\}$, we denote its parent by $\overleftarrow{x}$, and its children by $(x^{(1)}, \cdots, x^{(\mathtt{b}_x)})$. Write $|x|$ for the generation of x. Instead of looking at $\omega(x, y)$ (for $y \sim x$ and $x \in \mathbb{T}$), it is more convenient to use the notation

$$A(x) := \frac{\omega(\overleftarrow{x}, x)}{\omega(\overleftarrow{x}, \overleftarrow{\overleftarrow{x}})}, \qquad |x| \geq 2.$$

Recurrence/transience criteria

Hypothesis :

We assume that for all $|x| \geq 2$, $\{A(x^{(1)}), ..., A(x^{(\mathtt{b}_x)})\}$ has the same law as the vector $\{A_1, ..., A_{\mathtt{b}}\}$, where $\mathtt{b}_x$ denotes the number of children of x and $\mathtt{b}$ may be random. Define

$$\phi(t) := \log \mathbb{E}\Big(\sum_{i=1}^{\mathtt{b}} A_i^t \Big), \qquad \forall t \in \mathbb{R}.$$

Lyons and Pemantle (1992)'s theorem :

1. if $\inf_{0\leq t\leq 1}\phi(t) > 0$, then RWRE (X_n) is a.s. transient.
2. If $\inf_{0\leq t\leq 1}\phi(t) = 0$, then RWRE (X_n) is a.s. recurrent.
3. If $\inf_{0\leq t\leq 1}\phi(t) < 0$, then (X_n) is a.s. positive recurrent.

Case $\inf_{0\leq t\leq 1} \phi(t) = 0$

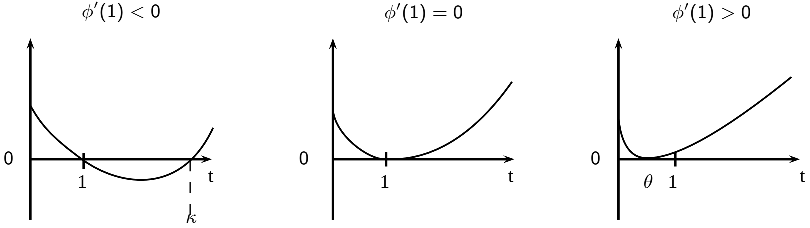

FIGURE: Three different shapes of ϕ : We call "subdiffusive case" the first shape and "slow movement case" the two other shapes.

Subdiffusive case

Theorem (H. and Shi 2007)

If $\inf_{0\le t\le 1}\phi(t)=0$ and $\phi'(1)<0$, then almost surely,

$$\max_{0\le i\le n}|X_i| = n^{\nu+o(1)},$$

where

$$\nu := 1-\max(\frac{1}{2},\frac{1}{\kappa}) \in (0,\frac{1}{2}],$$

and

$$\kappa := \inf\{t>1 : \phi(t)=0\} \in (1,\infty].$$

Slow movement case

Theorem (Faraud, H. and Shi 2012)

If $\inf_{0\le t\le 1}\phi(t)=0$ and $\phi'(1)\ge 0$, then almost surely,

$$\lim_{n\to\infty}\frac{1}{\log^3 n}\max_{0\le i\le n}|X_i| = c,$$

where

$$c := \begin{cases} \frac{8}{3\pi^2\phi''(1)}, & \text{if } \phi'(1)=0\,; \\ \frac{2\theta}{3\pi^2\phi''(\theta)}, & \text{if } \phi'(1)>0, \end{cases},$$

where $\theta\in(0,1]$ denotes the unique zero :$\phi'(\theta)=0$.

References

- E. Aïdékon (2008a&b) for rate of convergence and large deviations (transient case).
- Ben Arous and Hammond (2012), Hammond (2013) [subcritical/critical trees, stable laws].
- (sub)diffusive case ($\kappa > 2$) : is there an invariance principle to (reflected) Brownian motion ? Faraud (2011) confirms it for $\kappa > 5$. See recent work by E. Aïdékon and his PhD student Loïc de Raphélis.
- Andreoletti and Debs (2011+, 2013+) for the local times.
- If the ω are non random and $\mathbb{T}$ is a Galton-Watson tree, the model corresponds to the so-called biased random walk on (Galton-Watson) trees (see Peres and Zeitouni (2008) for a CLT in the recurrent case, Aïdékon (2013) for a formula on the speed in the transient case).

Boundary case : $\phi(1) = \phi'(1) = 0$

Call "boundary case"

When $\phi(1) = \phi'(1) = 0$, the associated potential process V is a branching random walk in the "boundary case", where

$$V(x) := -\sum_{y \in \rrbracket \varnothing, x \rrbracket} \log \frac{\omega(\overleftarrow{y}, y)}{\omega(\overleftarrow{y}, \overleftarrow{\overleftarrow{y}})} = -\sum_{y \in \rrbracket \varnothing, x \rrbracket} \log A(y).$$

Boundary case : $\phi(1) = \phi'(1) = 0$

Fact :
Recalling that $\frac{\max_{0 \le i \le n} |X_i|}{(\log n)^3} \to c$, a.s.

Open Problem :
What is the typical behavior of $|X_n|$? Can we localize X_n à la Sinai ?

Conjecture :
$\frac{|X_n|}{(\log n)^2}$ converges in law to a positive and finite random variable.

Boundary case : $\phi(1) = \phi'(1) = 0$

Fact :

Recalling that $\frac{\max_{0 \le i \le n} |X_i|}{(\log n)^3} \to c$, a.s.

Open Problem :

What is the typical behavior of $|X_n|$? Can we localize X_n à la Sinai ?

Conjecture :

$\frac{|X_n|}{(\log n)^2}$ converges in law to a positive and finite random variable.

Boundary case : $\phi(1) = \phi'(1) = 0$

Fact :

Recalling that $\frac{\max_{0 \le i \le n} |X_i|}{(\log n)^3} \to c$, a.s.

Open Problem :

What is the typical behavior of $|X_n|$? Can we localize X_n à la Sinai ?

Conjecture :

$\frac{|X_n|}{(\log n)^2}$ converges in law to a positive and finite random variable.

Potential energy

Definition

According to Le Doussal - Monthus (2002), $V(X_n)$ is called potential energy for the RWRE.

Motivation

The potential energy $V(X_n)$ is closely related

- to the localization of X_n,
- and to the Metropolis algorithm (Aldous (1998)), the Einstein relation on trees (Maillard and Zeitouni (2013+)).

Main result

Our main result on the maximal potential energy reads as follows :

Theorem 1 (H. and Shi'14+)

Assume $\phi(1) = \phi'(1) = 0$. Conditioned on $\{\mathbb{T} = \infty\}$,

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} \max_{0\le k\le n} V(X_k) = \frac{1}{2}, \qquad \mathbb{P}(d\omega) \otimes P_\omega\text{-a.s.}$$

Conjecture

Conditioned on $\{\mathbb{T} = \infty\}$, $\frac{1}{\log n} V(X_n)$ converges in law to a non-degenerated limit.

Main result

Our main result on the maximal potential energy reads as follows :

Theorem 1 (H. and Shi'14+)

Assume $\phi(1) = \phi'(1) = 0$. Conditioned on $\{\mathbb{T} = \infty\}$,

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} \max_{0\le k\le n} V(X_k) = \frac{1}{2}, \qquad \mathbb{P}(d\omega) \otimes P_\omega\text{-a.s.}$$

Conjecture

Conditioned on $\{\mathbb{T} = \infty\}$, $\frac{1}{\log n} V(X_n)$ converges in law to a non-degenerated limit.

The one-dimensional case : (RWRE (Z_n) on $\mathbb{Z}$)

1. By reversibility,

$$\limsup_{n\to\infty} \frac{1}{\log n} \max_{0\le k\le n} V(Z_k) \le 1, \qquad a.s.$$

   In fact, for any x, $P_{0,\omega}(T_x < n) \le n e^{-V(x)}$ where $T_x := \inf\{j > 0 : Z_j = x\}$. Choose x as the first such that $V(x) > (1+\varepsilon)\log n$, then $P_\omega(\max_{0\le k\le n} V(Z_k) > (1+\varepsilon)\log n) \le n^{-\varepsilon}$. Borel-Cantelli' lemma and the monotonicity yield the upper bound.

2. The vector

$$\left(\frac{V(Z_n)}{\log n}, \frac{\max_{0\le k\le n} V(Z_k)}{\log n}\right)$$

   converges in law to a non-degenerated limit ;

Proof of Theorem 1

- We can prove that a.s. on the survival set $\{\mathbb{T} = \infty\}$,

$$L_n^{\varnothing} = n^{1+o(1)},$$

where $L_n^{\varnothing}$ denotes the local time at $\varnothing$.

- By the standard extreme value theory, it is enough to prove the following statement : Almost surely on $\{\mathbb{T} = \infty\}$,

$$P_\omega\Big(\max_{0\le k\le T_\varnothing} V(X_k) \ge r\Big) = e^{-(1+o(1))\sqrt{2r}}, \qquad r \to \infty,$$

where $T_\varnothing := \inf\{n \ge 1 : X_n = \varnothing\}$.

First attempt to $e^{-\sqrt{2r}}$

- Notice that

$$P_\omega\Big(\max_{0\le k\le T_\varnothing} V(X_k)\ge r\Big)=P_\omega\Big(T_{\mathbb{H}_r}<T_\varnothing\Big),$$

where $\mathbb{H}_r := \Big\{x : V(x)\ge r, \max_{\varnothing\le y<x} V(y)<r\Big\}$ and $T_{\mathbb{H}_r} := \inf\{n\ge 1 : X_n\in\mathbb{H}_r\}$.

- Since $T_{\mathbb{H}_r}=\min_{x\in\mathbb{H}_r} T_x$, we get that

$$P_\omega\Big(T_{\mathbb{H}_r}<T_\varnothing\Big)\le\sum_{x\in\mathbb{H}_r}P_\omega\Big(T_x<T_\varnothing\Big)\le\sum_{x\in\mathbb{H}_r}e^{-V(x)}.$$

- Since $V(x)\to\infty$ a.s., we can add an additional condition that $\underline{V}(x)\ge-\alpha$ with some large constant α and $\underline{V}(x) := \min_{\varnothing\le y\le x} V(y)$.

First attempt to $e^{-\sqrt{2r}}$

By the many-to-one formule (Chauvin, Rouault and Wakolbinger (1991), Lyons, Pemantle and Peres (1995), Biggins and Kyprianou (2004)) at the stopping line $\mathbb{H}_r$, we obtain that

$$\mathbb{E}\Big[\sum_{x\in\mathbb{H}_r} e^{-V(x)}1_{\{\underline{V}(x)\geq-\alpha\}}\Big] = \mathbb{P}\Big(T_r < T_{-\alpha}\Big) \approx \frac{1}{r},$$

where $T_r := \inf\{n \geq 1 : S_n \geq r\}$, $T_{-\alpha} := \inf\{n \geq 1 : S_n \leq -\alpha\}$, and S is a centered real-valued random walk with finite variance. The bound $\frac{1}{r}$ is too big with respect to $e^{-\sqrt{2r}}$!

Second attempt to $e^{-\sqrt{2r}}$

- Define $V^{\#}(x) := \max_{\varnothing \le y \le x}(\overline{V}(y) - V(y))$ with $\overline{V}(y) := \max_{\varnothing \le z \le y} V(z)$.
- For $\lambda > 0$, we define another stopping line :

$$\mathbb{L}_\lambda^{\#} := \Big\{x : V^{\#}(x) > \lambda, \max_{\varnothing \le y < x} V^{\#}(y) \le \lambda\Big\}.$$

- On the event $\{T_{\mathbb{H}_r} < T_\varnothing\}$, the walk (X_n) hits some $x \in \mathbb{H}_r$ before return to the root, there are two cases : either $V^{\#}(x) > \lambda$, the walk (X_n) must have hit $\mathbb{L}_\lambda^{\#}$ before $T_\varnothing$; or $V^{\#}(x) \le \lambda$.

Second attempt to $e^{-\sqrt{2r}}$

- Then

$$P_\omega\Big(T_{\mathbb{H}_r} < T_\varnothing\Big) \le P_\omega\Big(T_{\mathbb{L}_\lambda^\#} < T_\varnothing\Big) + \sum_{x\in\mathbb{H}_r, V^\#(x)\le\lambda} P_\omega(T_x < T_\varnothing).$$

- In the same way as before,

$$P_\omega\Big(T_{\mathbb{L}_\lambda^\#} < T_\varnothing\Big) \le \sum_{x\in\mathbb{L}_\lambda^\#} P_\omega(T_x < T_\varnothing) \le \sum_{x\in\mathbb{L}_\lambda^\#} e^{-\overline{V}(x)} \le O(e^{-\lambda}),$$

because for $x \in \mathbb{L}_\lambda^\#$, $\overline{V}(x) - V(x) > \lambda$.

Second attempt to $e^{-\sqrt{2r}}$

- Since $P_\omega(T_x < T_\varnothing) \leq e^{-V(x)}$, we have by the many-to-one formula that

$$\mathbb{E}\Big[\sum_{x\in\mathbb{H}_r, V^\#(x)\leq\lambda} P_\omega(T_x < T_\varnothing)\Big] \leq \mathbb{P}\Big(S^\#_{T_r} \leq \lambda\Big).$$

- An a priori estimate for the random walk S :

$$\mathbb{P}\Big(S^\#_{T_r} \leq \lambda\Big) \approx e^{-r/\lambda}.$$

- Hence

$$\mathbb{E}\Big[P_\omega\Big(T_{\mathbb{H}_r} < T_\varnothing\Big)\Big] \leq O(e^{-\lambda}) + e^{-r/\lambda} = O(e^{-\sqrt{r}}),$$

if we take $\lambda = \sqrt{r}$.

Third (and last) attempt : from $e^{-\sqrt{r}}$ to $e^{-\sqrt{2r}}$

- Let $k = r^{1-\chi}$ with some $\frac{1}{2} < \chi < 1$. Cut the interval $[0, r]$ to k intervals $[h_m, h_{m+1}]$ for $h_m := r\frac{m}{k}$.
- Let $\lambda_m := \sqrt{r}g(1 - \frac{m}{k})$ for $0 \le m \le k$, with some positive function g to be optimized.
- Define $V_m^{\#}$ in the same way as $V^{\#}(x)$ but with those x between $\mathbb{H}_{h_m}$ and $\mathbb{H}_{h_{m+1}}$.

A refined argument to get $e^{-\sqrt{2r}}$

We have two cases :

- either for all $0 \le m < k$, $V_m^{\#} \le \lambda_m$, then

$$P_\omega\Big(T_{\mathbb{H}_r} < T_\varnothing\Big) \le \prod_{m=0}^{k-1} \mathbb{P}\Big(S^{\#}_{h_{m+1}-h_m} \le \lambda_m\Big) \approx e^{-\sum_{m=0}^{k-1} \frac{h_{m+1}-h_m}{\lambda_m}};$$

- or let m be the first one such that $V_m^{\#} > \lambda_m$, then

$$P_\omega\Big(T_{\mathbb{H}_r} < T_\varnothing\Big) \le e^{-\sum_{i=0}^{m-1} \frac{h_{i+1}-h_i}{\lambda_i}} e^{-\lambda_m}.$$

A refined argument to get $e^{-\sqrt{2r}}$

- Therefore

$$P_\omega\Big(T_{\mathbb{H}_r} < T_\varnothing\Big) \le e^{-\sum_{i=0}^{k-1} \frac{h_{i+1}-h_i}{\lambda_i}} + \sum_{m=1}^{k-1} e^{-\sum_{i=0}^{m-1} \frac{h_{i+1}-h_i}{\lambda_i}} e^{-\lambda_m}.$$

- With the choice $h_{i+1} - h_i = \frac{r}{k}$, $\lambda_i := \sqrt{r} g(1 - \frac{i}{k})$, we let $\lambda_m = \sum_{i=m}^{k-1} \frac{h_{i+1}-h_i}{\lambda_i} \sim \sqrt{r} \int_{m/k}^{1} \frac{du}{g(1-u)}$. Hence $g(u) = \sqrt{2u}$.
- We obtain the desired upper bound :

$$P_\omega\Big(T_{\mathbb{H}_r} < T_\varnothing\Big) \le e^{-\sqrt{2r}(1+o(1))}.$$

THANK YOU !