Asymptotic growth of a branching random walk in a random environment on the hypercube

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Model and some motivation

- n-dimensional hypercube $H_n = \{0,1\}^n$ with volume $N = 2^n$
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  $\rightarrow$ represents the 'fitness landscape'
- Branching random walk on H_n evolves as follows:
  - each particle jumps at rate κ to a uniformly chosen neighbour
  - a particle at x splits into two particles at rate $\xi(x)$

  $\rightarrow$ models the evolution of the population

The random environment and the objective

We are interested in, given the environment ξ,

$$E_{x_{k,N}}[\# \text{ particles alive at time } t] \sim \, ? \quad \text{as } t, n \to \infty$$

where $x_{k,N}$ is site with the kth highest branching rate, defined via

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Note, as $n \to \infty$,

- $\xi_{1,N} \sim \log(N) = n \log 2$
- for any fixed k, $\xi_{1,N} - \xi_{k,N} = const.$

The result on the complete graph

Branching random walk on the complete graph with N vertices

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Theorem (Fleischmann, Molchanov, '90)

Write $N_t = \#$ particles alive at time t.

(i) *if $k = 1$ or, if $k \geq 2$ and $t \ll \log N$,*

$$E_{x_{k,N}}[N_t] \sim e^{\lambda_{k,N} t},$$

(ii) *if $k \geq 2$ and $t \gg \log N$,*

$$E_{x_{k,N}}[N_t] \quad \sim \quad \kappa[N(\xi_{1,N} - \xi_{k,N})]^{-1} \; e^{\lambda_{1,N} t},$$

where $\lambda_{k,N}$ is the kth eigenvalue of $\kappa \Delta_N + \xi_N I_N$.

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- $\lambda_{k,N} = \xi_{k,N} - \kappa + \kappa/N + o(1/N)$.
- $E_{x_{k,N}}[N_t] = \mathbb{E}_{x_{k,N}}[\exp\{\int_0^t \xi(X_s)ds\}]$, where $X = (X_s)_{s\geq 0}$ is random walk on the complete graph.

The result on the hypercube

Theorem (Avena, Gün, H., '14)

(i) *If* $t \ll n \log n$ $(\approx logN \log\log N)$,

$$E_{x_{k,N}}[N_t] \sim e^{(\xi_{k,N}-\kappa)t},$$

(ii) *If* $t \gg n \log n$,

$$\begin{aligned} \log E_{x_{k,N}}[N_t] &\sim \log E_{x_{1,N}}[N_t] \\ \textit{and} \quad E_{x_{1,N}}[N_t] &\sim e^{(\xi_{1,N}-\kappa+\frac{\kappa^2}{n^2\log 2}+O(n^{-5}))t}. \end{aligned}$$

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- $t \ll n \log n$: X stays at $x_{k,N}$ since

$$E_{x_{k,N}}[N_t] = \mathbb{E}_{x_{k,N}}[\exp\{\int_0^t \xi(X_s)ds\}; X_s = x_{k,N} \; s \leq t] = e^{\xi_{k,N}t}e^{-\kappa t}$$

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- $n \log n \ll t \ll n^2$: X goes to $x_{1,N}$ and stays there
- $n^2 \ll t \ll n^5$: X goes to $x_{1,N}$ and then stays within dist. 1.

Bounds required for the phase transition

For $k \geq 2$, study

$$(\star) \quad = \quad \lim_{t,n\to\infty} \frac{\mathbb{E}_{x_{k,N}}[e^{\int_0^t \xi(X_s)ds}; X_s = x_{1,N}, \text{ for some } s \leq t]}{\mathbb{E}_{x_{k,N}}[e^{\int_0^t \xi(X_s)ds}; X_s = x_{k,N}, \text{ for all } s \leq t]}$$

$$= \quad \lim_{t,n\to\infty} e^{-(\xi_{K,N}-\kappa)t}\mathbb{E}_{x_{k,N}}[e^{\int_0^t \xi(X_s)ds}; X_s = x_{1,N}, \text{ for some } s \leq t]$$

and show that

- if $t \ll n \log n$, then $(\star) \leq 0$
- if $t \gg n \log n$, then $(\star) \geq \infty$

The lower bound for $t \gg n \log n$

Note: $d(x_{k,N}, x_{1,N}) \sim n/2$, as $n \to \infty$ (d is Hamming distance)

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Let γ be path that goes from $x_{k,N}$ to $x_{1,N}$ in exactly $n/2$ steps in time s and stays at $x_{1,N}$ up to time t.

- $P_{x_{k,N}}(\text{go to } x_{1,N} \text{ in exactly } n/2 \text{ steps}) = \frac{(\frac{n}{2})!}{n^{n/2}}$
- $P(\text{make } n/2 \text{ jumps by time } s) = \frac{(\kappa s)^{n/2}}{(\frac{n}{2})!} e^{-\kappa s}$

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Thus

$$\mathbb{E}_{x_{k,N}}[e^{\int_0^t \xi(X_s) ds}, X = \gamma] \quad \geq \quad \frac{(\frac{n}{2})!}{n^{n/2}} \frac{(\kappa s)^{n/2}}{(\frac{n}{2})!} e^{-\kappa s} e^{(\xi_{1,N} - \kappa)(t-s)}$$

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and 'cost' is minimised for $s = 1/(2 \log 2)$.

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Thanks!