Contraction of trees

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based on joint work with

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Problem statement (1)

$T = (V, E, \rho)$ random rooted tree (in the graph theoretic sense), locally finite. For $p \in (0,1)$, define the random tree $\mathcal{C}_p(T)$ by *contracting* each edge in T with probability $1-p$. Contracting an edge means removing it and identifying its head and tail.

Equivalent definition: $V' =$ set containing each vertex with probability p (plus root). Construct tree on V' by preserving ancestral relationships.

Note: Resulting tree need not be locally finite (if the critical point p_c of edge percolation on the tree satisfies $1-p > p_c$)

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Characterize/construct all p-self-similar trees.

Large body of literature concerning dynamics on random trees:

- Growth - construction of consistent families of uniform trees with n leaves (Rémy (1985), Aldous (1991), Marchal (2008) ...)
- Percolation on the edges: different perspective however - dynamic of the component containing the root (Aldous and Pitman (1998), Miermont (2005), Abraham and Delmas (2012)...)
- Subtree pruning and regrafting (Evans and Winter (2006),...)
- Percolation on the leaves (Duquesne and Winkel (2007), ...)

Trivialities

Problem

Characterize/construct all p-self-similar trees.

Necessary conditions for T to be self-similar:

- T is infinite
- Finite number of infinite rays, separating at root.

Trivial examples of p-self-similar trees: $\mathbb{N}$, $\mathbb{N} \sqcup \ldots \sqcup \mathbb{N}$.

Further examples.

First example

Bouquet example

Attach to each vertex of $\mathbb{N}$ a bouquet of edges, with independent $\mathrm{Geo}(q)$ number of edges edges in each bouquet, $q \in (0, 1]$.

Relies on the claim: take $F \sim \mathrm{Geo}(p)$, and $G_i \sim \mathrm{Geo}(q)$ independent, the sum of $F + 1$ independent Binomial random variables with parameters G_i and p has law G_1.

$$\sum_{i=1}^{F+1} B_i \stackrel{(d)}{=} G_1, \text{ with } B_i \text{ iid } Bin(G_i, p)$$

To *find* this: look for the stationary distribution of the continuous time Markov process N

- $N \to N + N'$ at rate 1, with N' an independent copy of N,
- $N \to N - 1$ at rate N

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To *check* this:

- Take two independent Poisson processes on the real line, with intensity i_1 and i_2.
- Number of points of the second before the first point of the first is Geo$(i_1/(i_1 + i_2))$.
- Thin the two processes with the same parameter p.

Real trees

A *real tree* is a geodesic metric space $(\mathcal{V}, d)$ "without cycles".

Definition

An $\mathbb{R}$-tree is a metric space $\mathcal{T} = (\mathcal{V}, d)$ with the following properties:

1. It is *geodesically linear*, i.e. for every $x, y \in \mathcal{V}$, there is a unique isometry $f_{x,y} : [0, d(x, y)] \to \mathcal{V}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x, y)) = y$.
2. It is "without loops", i.e. for every $x, y \in \mathcal{V}$, if r and q are continuous injective maps from $[0, 1]$ to $\mathcal{V}$ such that $q(0) = x$ and $q(1) = y$, and $r(0) = x$ and $r(1) = y$, then $q([0, 1]) = r([0, 1])$.

The *length measure* on $\mathcal{T} = (\mathcal{V}, d)$ is a σ-finite measure $\ell_{\mathcal{T}}$ on $\mathcal{V}$ that satisfies:

$$\forall a, b \in \mathcal{V} : \ell_{\mathcal{T}}(]a, b[) = d(a, b)$$

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Definition

- $\mathfrak{T}$: space of (equivalence classes of) *measured, rooted, real, locally compact trees* $\mathcal{T} = (\mathcal{V}, d, \rho)$ with a locally finite length measure ℓ.
- $\mathfrak{T}_1 \subset \mathfrak{T}$ the subspace where ℓ is a probability measure.

We endow these trees with the *Gromov–Hausdorff (GH) topology*.

Note: Most prominent examples of real trees do not have a locally finite length measure, e.g. Aldous’ *(Brownian) continuum random tree*.

Rescaling /discretization.

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Rescaling: For $\mathcal{T} = (\mathcal{V}, d, \rho) \in \mathfrak{T}$ and $p > 0$, we define the rescaled tree $\mathcal{S}_p(\mathcal{T})$ by

$$\mathcal{S}_p(\mathcal{T}) = (\mathcal{V},\ p \cdot d,\ \rho).$$

Definition

We say a (random) tree $\mathcal{T}$ taking values in $\mathfrak{T}$ is p-self-similar, $p \in (0, 1)$, if $\mathcal{T}$ and $\mathcal{S}_p(\mathcal{T})$ are equal in law (up to measure-preserving isometries fixing the root).

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Discretization: For $\mathcal{T} = (\mathcal{V}, d, \rho, \mu) \in \mathfrak{T}$, we define the discretized tree $\mathcal{D}(\mathcal{T})$ as follows: Sample a random set of vertices $V_0 \subset \mathcal{V}$ according to a Poisson process with intensity $\ell_{\mathcal{T}}$. Then $\mathcal{D}(\mathcal{T})$ is the discrete tree with the following properties:

- The set of vertices is $V = \{\rho\} \cup V_0$,
- For two vertices $v, w \in V$,

$$v \preceq_{\mathcal{D}(\mathcal{T})} w \iff v \preceq_{\mathcal{T}} w.$$

($v \preceq_{\mathcal{T}} w$ if v lies on geodesic between ρ and w in $\mathcal{T}$)

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Commutation relation

For every $p \in (0, 1)$,

$$\mathcal{C}_p \circ \mathcal{D} = \mathcal{D} \circ \mathcal{S}_p.$$

Statement

Second example

If $\mathcal{T} \in \mathfrak{T}$ is a random real p-self-similar tree, then the discrete random tree T given by

$$T = \mathcal{D}(\mathcal{T})$$

is p-self-similar again.

Proof is elementary:

$$\mathcal{C}_p \circ \mathcal{D}(\mathcal{T}) = \mathcal{D} \circ \mathcal{S}_p(\mathcal{T}) = \mathcal{D}(\mathcal{T}) \text{ when } \mathcal{T} \text{ is p-self similar.}$$

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What about the converse? Is the bouquet example contained in this example?

Towards the converse

Iterating the contraction n times yields:

$$T = \mathcal{C}_p(T) = \mathcal{C}_p \circ \ldots \mathcal{C}_p(T) = \mathcal{C}_{p^n}(T).$$

With ιT is the embedding of the discrete tree T into the space of real trees $\mathfrak{T}$ - just adding edge length 1 between two adjacent vertices:

$$\mathcal{C}_{p^n}(T) = \mathcal{D}(\mathcal{S}_{p^n}(\iota T)) \text{ w.h.p.}$$

Then, find the appropriate subspace of the set of real trees in which the sequence of rescaled real trees $\mathcal{S}_{p^n}(\iota T)$ is tight, and the map $\mathcal{D}$ is continuous. Then, if $\mathcal{T}$ is a limit point,

$$T = \mathcal{D}(\mathcal{T}).$$

Towards the converse

Now, we can devise the tree $\mathcal{T}$ associated with the bouquet example (in which a Geo(q) number of edges is attached to each vertex of $\mathbb{N}$). It is a random real measured tree, that is the limit of the sequence of rescaled (real) trees $\mathcal{S}_{p^n}(\iota T)$. If ℓ denotes the Lebesgue measure,

$$\mathcal{T} = (\mathbb{R}_+, d_{\mathrm{Eucl}}, 0, (1 + \frac{1-q}{q}) \cdot \ell).$$

and we have to distinguish whether a point is sampled according to ℓ or $\frac{1-q}{q} \cdot \ell$ in the discretization.

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So we need to:

- redefine the state space,
- and the discretization operation on it.

Characterization.

Real trees

Definition

- $\mathfrak{T}$: space of (equivalence classes of) *measured, rooted, real, locally compact trees* $\mathcal{T} = (\mathcal{V}, d, \rho, \mu)$ where μ is a locally finite measure,
- $\mathfrak{T}_1 \subset \mathfrak{T}$ the subspace where μ is a probability measure,
- $\mathfrak{T}^\ell \subset \mathfrak{T}$ and $\mathfrak{T}_1^\ell \subset \mathfrak{T}_1$ the subspaces where $\mu \geqslant \ell_{\mathcal{T}}$.

We endow these trees with the *Gromov–Hausdorff–Prokhorov topology*, which makes $\mathfrak{T}$ topologically complete (ADH13).

Rescaling/Discretization

We re-define the *discretization/Poissonian sampling* in a consistent way.

Rescaling: For $\mathcal{T} = (\mathcal{V}, d, \rho, \mu) \in \mathfrak{T}^{\ell}$ and $p > 0$, we define the rescaled tree $\mathcal{S}_p(\mathcal{T})$ by

$$\mathcal{S}_p(\mathcal{T}) = (\mathcal{V},\, p \cdot d,\, \rho,\, p \cdot \mu).$$

Definition

We say a (random) tree $\mathcal{T}$ taking values in $\mathfrak{T}^{\ell}$ is p-self-similar, $p \in (0, 1)$, if $\mathcal{T}$ and $\mathcal{S}_p(\mathcal{T})$ are equal in law (up to measure-preserving isometries fixing the root).

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Discretization: For $\mathcal{T} = (\mathcal{V}, d, \rho, \mu) \in \mathfrak{T}^{\ell}$, we define the discretized tree $\mathcal{D}(\mathcal{T})$ as follows: Sample two random (multi-)sets of vertices $V_0, V_1 \subset \mathcal{V}$ according to independent Poisson processes with intensity $\ell_{\mathcal{T}}$ and $\mu - \ell_{\mathcal{T}}$, respectively. Then $\mathcal{D}(\mathcal{T})$ is the discrete tree with the following properties:

- The set of vertices is $V = \{\rho\} \cup V_0 \cup V_1$,
- For two vertices $v, w \in V$,

$$v \preceq_{\mathcal{D}(\mathcal{T})} w \iff v \preceq_{\mathcal{T}} w \text{ and } v \in V_0 \cup \{\rho\}.$$

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Main result

Theorem

There exists a one-to-one correspondence between

- random discrete p-self-similar trees T and
- random real p-self-similar trees $\mathcal{T}$ taking values in $\mathfrak{T}^{\ell}$,

given by

$$T = \mathcal{D}(\mathcal{T}).$$

Elements of the proof

- We already explained how $\mathcal{T}$ arises as the limit of the sequence of rescaled trees $\mathcal{S}_{p^n}(\iota T)$ in the GHP topology.
- For the uniqueness, we rely on the injectivity of the discretization map $\mathcal{D}$ which itself is a consequence of the Gromov-Vershik characterization of measured metric spaces through their distance matrix distribution

$$(D_{ij})_{i,j\in\mathbb{N}} = (d(X_i, X_j))_{i,j\in\mathbb{N}}$$

  where $X_1, X_2, \ldots$ are iid according to μ.
- Last, $\mathcal{T}$ inherits from T the self-similarity property:

$$\mathcal{S}_p(\mathcal{T}) = \lim_{n\to\infty} \mathcal{S}_{p^{n+1}}(\iota T) = \lim_{n\to\infty} \mathcal{S}_{p^n}(\iota T) = \mathcal{T}.$$

p-self-similar real trees

Examples of p-self-similar real trees

Construction through subordination of a real-valued self-similar process. Ingredients:

1. A random real tree $\mathcal{T}_0$ taking values in $\mathfrak{T}_1^\ell$.
2. A real-valued process $(X(t); t \geqslant 0)$, which is *increasing*, *pure-jump* and satisfies

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Construct a p-self-similar real tree as follows:

- Start with an infinite ray (the spine).
- For each jump time t of the process X, take an independent copy $\mathcal{T}_0^{(t)}$ of $\mathcal{T}_0$, and attach its rescaling $S_{X(t)-X(t-)}(\mathcal{T}_0^{(t)})$ to the spine at distance t from the root.

Translation invariant trees

Question

Can one construct examples of one-ended p-self-similar trees $\mathcal{T} = (\mathcal{V}, d, \rho, \mu)$ which are *translation invariant* (in law) along the spine?

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Denote by v_t the spine vertex at distance t from the root and by $\mathcal{V}^{\leqslant t}$ the subset of vertices which are not descendants of v_t. Define the *mass process* $(X(t); t \geqslant 0)$ by $X(t) = \mu(\mathcal{V}^{\leqslant t})$. Then $(X(t); t \geqslant 0)$ is a real-valued, increasing, stochastic process with stationary increments satisfying,

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Theorem (basically Vervaat (1985))

Let $(X(t); t \geqslant 0)$ be a process as above. Then, almost surely, for every $t \geqslant 0$, $X(t) = X(1)\, t$.

Translation invariant trees (2)

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Corollary

A random, one-ended tree $\mathcal{T}$ taking values in $\mathfrak{T}^{\ell}$, which is translation invariant along the spine, is p-self-similar if and only if

$$\mathcal{T} = (\mathbb{R}_+, d_{\mathrm{Eucl}}, 0, Y \cdot \ell), \quad Y \geqslant 1 \text{ a random variable.}$$

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$$\mathcal{T} = (\mathbb{R}_+, d_{\mathrm{Eucl}}, 0, Y \cdot \ell), \quad Y \geqslant 1 \text{ a random variable.}$$

Corollary

The unique random, one-ended, translation invariant and p-self-similar discrete tree T is the bouquet example.

A generalization

To get more interesting examples, generalize the contraction and rescaling operations $\mathcal{C}_p$ and $\mathcal{S}_p$: Let $p, q \in (0, 1)$.

- $\mathcal{C}_{p,q}$: Defined as $\mathcal{C}_p$, but vertices on the spine are retained with probability q.
- $\mathcal{S}_{p,q}$: Defined as $\mathcal{S}_p$, but distances on the spine are rescaled by q instead of p.

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Definition

A random (discrete) T is (p, q)-*self-similar* if $T \overset{\text{law}}{=} \mathcal{C}_{p,q}(T)$.
A random (real) tree $\mathcal{T}$ is (p, q)-*self-similar* if $\mathcal{T} \overset{\text{law}}{=} \mathcal{S}_{p,q}(\mathcal{T})$.

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The characterization holds with p-self-similar replaced by (p, q)-self-similar.

The iid case

In the translation invariant case, many examples can be constructed when $q > p$. Let us consider the case where the subtrees along the spine are iid. Write the (discrete) tree T as $T = (T^0, T^1, \dots)$, where T^n is the subtree of the n-th vertex on the spine. We construct a (p, q)-self-similar tree where $T^0, T^1, \dots$ are iid. The ingredients are the following:

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- $(\mathcal{T}_0^n)_{n \geqslant 0}$: an iid sequence of trees in $\mathfrak{T}_1^\ell$
- ν: a quasi-stationary distribution with eigenvalue q of the Galton–Watson process $(Z_n; n \geqslant 0)$ with offspring distribution $p_0 = 1 - p$, $p_1 = p$. That is, ν satisfies

$$\forall n \in \mathbb{N} : \mathbb{P}_\nu(Z_n \in \cdot \,|\, Z_n > 0) = \nu \quad \text{and} \quad \mathbb{P}_\nu(Z_1 > 0) = q.$$

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- A constant $c \in (0, 1]$.

The iid case (2)

- $(\mathcal{T}_0^n)_{n\geqslant 0}$: an iid sequence of trees in $\mathfrak{T}_1^\ell$
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Construct tree $T = (T^0, T^1, \ldots)$, where $T^0, T^1, \ldots$ are iid according to the following law:

T^0 is the union of a Geo(c)-distributed number of iid trees T', where

$$T' \overset{\text{law}}{=} \mathcal{D}(\mathcal{T}_0, N), \quad N \sim \nu.$$

Here, $\mathcal{D}(\mathcal{T}_0, m)$ is the tree $\mathcal{D}(\mathcal{T}_0)$ cond'ed on having m vertices (plus root).

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”Theorem”: This example (almost) covers all cases.

Conclusion

- We characterized all (p, q)-self-similar trees in terms of certain *(measured) real trees* satisfying a simple (multiplicative) self-similarity property; these real trees arise as scaling limits of these trees.

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Conclusion

- We characterized all (p, q)-self-similar trees in terms of certain *(measured) real trees* satisfying a simple (multiplicative) self-similarity property; these real trees arise as scaling limits of these trees.
- We constructed several classes of examples of such trees.
- The limiting real trees have a locally finite finite length measure. As a consequence, the (p, q)-self-similar trees are rather elongated, very different from Galton–Watson trees (for example).

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- We characterized all (p, q)-self-similar trees in terms of certain *(measured) real trees* satisfying a simple (multiplicative) self-similarity property; these real trees arise as scaling limits of these trees.
- We constructed several classes of examples of such trees.
- The limiting real trees have a locally finite finite length measure. As a consequence, the (p, q)-self-similar trees are rather elongated, very different from Galton–Watson trees (for example).
- Usually in the literature, operations on trees act on the *leaves* of the trees or on whole subtrees, not on single internal vertices.

Thanks.