The scaling limit of the minimum spanning tree of the complete graph

Christina Goldschmidt
University of Oxford

Joint work with Louigi Addario-Berry (McGill), Nicolas Broutin (INRIA Rocquencourt) and Grégory Miermont (ENS Lyon).
Minimum spanning trees

Suppose that $G = (V(G), E(G))$ is a finite connected graph. To each edge $e \in E(G)$, associate a positive weight $w(e)$.

**Problem:** find the connected subgraph $T$ of $G$ which has the same vertex set as $G$ and minimises

\[
\sum_{e \in E(T)} w(e).
\]

Since the weights are positive, the minimum is necessarily attained by a tree. If the weights are all distinct, this tree is unique and is the so-called **minimum spanning tree (MST)**.
There are three commonly-used algorithms for finding the MST of a graph:

- Kruskal’s algorithm
- Prim’s algorithm
- Cycle-breaking (reverse Kruskal) algorithm.
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Kruskal’s algorithm

▶ Start from a forest of isolated vertices. List the edges as $e_1, e_2, \ldots$ in increasing order of weight.
▶ At step $i$, add edge $e_i$ as long as it does not create a cycle.
▶ Stop when all vertices are connected.

At every stage, we have a forest.
Kruskal’s algorithm
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Cycle-breaking algorithm

- Start with all edges present. List the edges as $e_1, e_2, \ldots$ in decreasing order of weight.
- At step $i$, remove $e_i$ as long as it does not disconnect the graph (i.e. as long as it lies in a cycle).
- Stop when no cycles remain.
Cycle-breaking algorithm
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![Diagram of cycle-breaking algorithm]
Cycle-breaking algorithm
Cycle-breaking algorithm

1.2

2.1

1.7

0.9

3.4

3.6

6.7

5.3

3.6

2.3

1.3
The minimum spanning tree of the complete graph

Consider the complete graph on \( n \) vertices with independent edge-weights which are uniformly distributed on \([0, 1]\).
The minimum spanning tree of the complete graph

Consider the complete graph on $n$ vertices with independent edge-weights which are uniformly distributed on $[0, 1]$. 
Question (Aldous (1990))

Does the MST of the complete graph on $n$ vertices possess a scaling limit?
We don’t really care about the vertex labels or the weights.
Graphs as metric spaces

So we will think of the tree as a metric space using the graph distance for the metric.
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So we will think of the tree as a **metric space** using the **graph distance** for the metric. We will also want to be able to pick points uniformly at random from the tree, so we equip it with the **uniform measure** (call this a **measured metric space**).
An easier scaling limit: the uniform spanning tree

Suppose that instead we consider $T_n$, the \textit{uniform} spanning tree (UST) of the complete graph on $n$ vertices.
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Again, think of $T_n$ as a measured metric space by using the graph distance and assigning each vertex weight $1/n$. 
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Again, think of $T_n$ as a measured metric space by using the graph distance and assigning each vertex weight $1/n$.

**Theorem.** (Aldous (1993); Le Gall (2006)) As $n \to \infty$,

$$\frac{1}{\sqrt{n}} T_n \overset{d}{\to} \mathcal{T},$$

in the sense of the Gromov-Hausdorff-Prokhorov distance $d_{\text{GHP}}$ between compact measured metric spaces, where $\mathcal{T}$ is the Brownian continuum random tree.
Real trees from excursions

Let $h : [0, 1] \rightarrow \mathbb{R}^+$ be an excursion, that is a continuous function such that $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in (0, 1)$.
Real trees from excursions

Now put glue on the underside of the excursion and push the two sides together...
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Now put glue on the underside of the excursion and push the two sides together to get a tree.
Real trees from excursions

Now put glue on the underside of the excursion and push the two sides together to get a real tree.
The Brownian CRT

The Brownian continuum random tree \( \mathcal{T} \) is the random real tree we obtain by doing this gluing procedure to the function \( 2e \), where \( (e(x), 0 \leq x \leq 1) \) a standard Brownian excursion.
Properties of the Brownian CRT

Since local minima of $e$ are a.s. unique, $\mathcal{T}$ is binary.
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$\mathcal{T}$ comes naturally equipped with a uniform measure $\mu$, which is the probability measure induced on $\mathcal{T}$ from the Lebesgue measure on $[0, 1]$. $\mu$ is concentrated on the leaves of the tree.
Consider $T_n$, the UST of the complete graph on $n$ vertices. Think of $T_n$ as a measured metric space by using the graph distance and assigning each vertex weight $1/n$.

**Theorem.** (Aldous (1993); Le Gall (2006)) As $n \to \infty$, 

$$\frac{1}{\sqrt{n}} T_n \xrightarrow{d_{\text{GHP}}} \mathcal{T},$$

in the sense of the Gromov-Hausdorff-Prokhorov distance $d_{\text{GHP}}$ between compact measured metric spaces, where $\mathcal{T}$ is the Brownian continuum random tree.
The distance between measured metric spaces

Suppose that \((X, d, \mu)\) and \((X', d', \mu')\) are measured metric spaces.
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A correspondence \(R\) is a subset of \(X \times X'\) such that for every \(x \in X\), there exists \(x' \in X'\) with \((x, x') \in R\) and vice versa.
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The distortion of \(R\) is

\[
\text{dis}(R) = \sup \{|d(x, y) - d'(x', y')| : (x, x'), (y, y') \in R\}.
\]
Measuring the distance between measured metric spaces

$(X, d, \mu)$ and $(X', d', \mu')$ are close in the Gromov-Hausdorff-Prokhorov distance if:

We can find a correspondence $R$ between $X$ and $X'$ and a measure $\pi$ on $X \times X'$ in such a way that
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- $R$ has small distortion (i.e. $d(x, y)$ and $d'(x', y')$ are close whenever $x \in X$ corresponds to $x' \in X'$ and $y \in X$ corresponds to $y' \in X'$);
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- $R$ has small distortion (i.e. $d(x, y)$ and $d'(x', y')$ are close whenever $x \in X$ corresponds to $x' \in X'$ and $y \in X$ corresponds to $y' \in X'$);
- $\pi$ is almost a coupling of $\mu$ and $\mu'$;
- $\pi(R^c)$ is small.
Consider $T_n$, the UST of the complete graph on $n$ vertices. Think of $T_n$ as a measured metric space by using the graph distance and assigning each vertex weight $1/n$.

**Theorem.** (Aldous (1993); Le Gall (2006)) As $n \to \infty$,

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\frac{1}{\sqrt{n}} T_n \xrightarrow{d} \mathcal{T},
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in the sense of the Gromov-Hausdorff-Prokhorov distance $d_{\text{GHP}}$ between compact measured metric spaces, where $\mathcal{T}$ is the Brownian continuum random tree.
Back to the original question: what about the MST?

Does the MST of the complete graph on $n$ vertices possess a scaling limit?
Hold on, isn’t it the same?!

Figure 4: A uniform random tree (left) and a minimum spanning tree (right) on 3000 vertices

This proof as well as its empirical illustration correspond to the fact that the distances on minimum spanning trees, as $n \to \infty$, grow by a factor of $n^{1/3}$, whereas the distances on uniform random trees grow faster, by a factor of $n^{1/2}$. Roughly speaking, uniform random trees tend to be more stretched than minimum spanning trees. This is also illustrated in Figure 4.

### 2.3 Illustration of Theoretical Results for Uniform Random Trees

Consider a uniform random tree on $n$ vertices, labelled $\{1, 2, \ldots, n\}$. As $n$ approaches infinity and distances of the tree are rescaled by $n^{-1/2}$, the tree converges to the so-called Brownian Continuum Random Tree. It satisfies the following conditions [2]:

1. The reduced tree on $k$ randomly chosen vertices is a binary tree with all chosen vertices on the leaves (almost surely). The tree shape of the $k$-reduced tree is uniformly distributed on the set of binary trees with $k$ labelled leaves,
2. The $2k - 3$ segmental lengths of the reduced tree follow a joint probability distribution.
The scaling limit of the MST

Let $M_n$ be the MST of the complete graph on $n$ vertices, considered as a metric space using the graph distance, and endowed with the uniform measure on its vertices.

**Theorem.** (Addario-Berry, Broutin, G. & Miermont) There exists a random compact measured metric space $\mathcal{M}$ such that

$$\frac{1}{n^{1/3}} M_n \stackrel{d}{\rightarrow} \mathcal{M}$$

as $n \to \infty$, in the sense of $d_{\text{GHP}}$. 

The key to understanding this result is a connection to the Erdős-Rényi random graph.
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The key to understanding this result is a connection to the Erdős-Rényi random graph.
Reminder: the Erdős-Rényi random graph

Take $n$ vertices labelled by $\{1, 2, \ldots, n\}$ and put an edge between any pair independently with probability $p$. Call the resulting model $\mathcal{G}(n, p)$. (Bond percolation on the complete graph.)

Example: $n = 10$, $p = 0.4$ (vertex labels omitted).
The Erdős-Rényi random graph process

Take the obvious coupling of \((\mathbb{G}(n, p), p \in [0, 1])\): simply assign each edge \(e\) an independent random edge-weight \(W(e)\) with uniform distribution on \([0, 1]\) and keep all edges with weight \(W(e) \leq p\).
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It’s then natural to think of \( p \) as a time and the evolving graph as a process.
Kruskal and Erdős-Rényi

By using the same weights, we can think of (a continuous-time version of) Kruskal’s algorithm as “sitting inside” the Erdős-Rényi process: we call this the **Kruskal process**. When an edge appears in the Erdős-Rényi process, it gets included in the Kruskal process as long as it doesn’t create a cycle.
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In particular, the components in both processes (which are all trees in the case of Kruskal) have the same vertex-sets.

Indeed, the Kruskal components are the MST’s of the Erdős-Rényi components.
The phase transition

Sitting inside each component of the Erdős-Rényi process is a component of the Kruskal process.
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So the action is happening a bit earlier, in the critical window for the Erdős-Rényi phase transition.
The critical random graph

The critical window: $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, where $\lambda \in \mathbb{R}$. For such $p$, the largest components have size $\Theta(n^{2/3})$. 

Define the surplus of a component to be the number of edges more than a tree that it has.
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A component with surplus 3:
Convergence of the sizes and surpluses

Fix $\lambda \in \mathbb{R}$. Let $C_{\lambda}^{n,1}, C_{\lambda}^{n,2}, \ldots$ be the sequence of component sizes of $\mathbb{G}(n, n^{-1} + \lambda n^{-4/3})$ in decreasing order, and let $S_{\lambda}^{n,1}, S_{\lambda}^{n,2}, \ldots$ be their surpluses.

Write $C_{\lambda}^{n} = (C_{\lambda}^{n,1}, C_{\lambda}^{n,1}, \ldots)$ and $S_{\lambda}^{n} = (S_{\lambda}^{n,1}, S_{\lambda}^{n,2}, \ldots)$. 

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**Theorem** (Aldous (1997)). For fixed $\lambda \in \mathbb{R}$, as $n \to \infty$,

$$(n^{-2/3} C^n_\lambda, S^n_\lambda) \xrightarrow{d} (C_\lambda, S_\lambda).$$
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Fix $\lambda \in \mathbb{R}$. Let $C_{\lambda}^{n,1}, C_{\lambda}^{n,2}, \ldots$ be the sequence of component sizes of $G(n, n^{-1} + \lambda n^{-4/3})$ in decreasing order, and let $S_{\lambda}^{n,1}, S_{\lambda}^{n,2}, \ldots$ be their surpluses.

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$$(n^{-2/3} C_{\lambda}^{n}, S_{\lambda}^{n}) \xrightarrow{d} (C_{\lambda}, S_{\lambda}).$$

(Convergence for the first co-ordinate takes place in

$$\ell^2_{\lambda} := \left\{ x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

and for the second in the sense of finite-dimensional distributions.)
Limiting sizes and surpluses

Let $W_\lambda(t) = W(t) + \lambda t - \frac{t^2}{2}$, $t \geq 0$, where $(W(t), t \geq 0)$ is a standard Brownian motion.
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Let $B_\lambda(t) = W_\lambda(t) - \min_{0 \leq s \leq t} W_\lambda(s)$ be the process reflected at its minimum.

[Pictures by Louigi Addario-Berry]
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Decorate the picture with the points of a rate one Poisson process which fall above the $x$-axis and below the graph.

$C_\lambda$ is the sequence of excursion-lengths of this process, in decreasing order.

$S_\lambda$ is the sequence of numbers of points falling in the corresponding excursions.
The critical Erdős-Rényi random graph

Let $G^{n}_{\lambda} = (G^{n,1}_{\lambda}, G^{n,2}_{\lambda}, \ldots)$ be the sequence of components of $G(n, n^{-1} + \lambda n^{-4/3})$ in decreasing order of size, each considered as a metric space with the graph distance. Put mass $n^{-2/3}$ on each vertex.
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**Theorem (A-B, B, G).** As \( n \to \infty \),

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n^{-1/3} G^n_\lambda \overset{d}{\to} G_\lambda,
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where \( G_\lambda = (G^1_\lambda, G^2_\lambda, \ldots) \) is a sequence of measured real trees with vertex-identifications encoded by Aldous' marked limit process in decreasing order of length.
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where $\mathcal{G}_\lambda = (\mathcal{G}^1_\lambda, \mathcal{G}^2_\lambda, \ldots)$ is a sequence of measured real trees with vertex-identifications encoded by Aldous’ marked limit process in decreasing order of length.

Convergence is with respect to the distance

$$\text{dist}^4_{\text{GHP}}(A, B) := \left( \sum_{i=1}^{\infty} d_{\text{GHP}}(A_i, B_i)^4 \right)^{1/4}.$$
The limit of the random graph

- Each excursion of the process \((B_\lambda(t), t \geq 0)\) encodes a random real tree, which is a "spanning tree" for a limit component.
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- The \(k\)th-longest excursion has length \(C_\lambda^k\) and so the corresponding tree is endowed with a uniform measure of that total mass.
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- In the limit, \textit{surplus edges correspond to vertex-identifications}. 
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- These are not scaled Brownian CRT’s, but random real trees whose distribution has been biased in favour of excursions with larger area.
- The \( k \)th-longest excursion has length \( C_\lambda^k \) and so the corresponding tree is endowed with a uniform measure of that total mass.
- In the limit, surplus edges correspond to vertex-identifications.
- In each excursion, the points of the Poisson process tell us where these vertex-identifications should occur.
Spanning tree and vertex identifications
To get from the state of the Erdős-Rényi process at a fixed time $p \in [0, 1]$ to the state of the Kruskal process, we need to break the cycles at high-weight edges.
Consider the Erdős-Rényi random graph at the point $\lambda$ in the critical window: $G^\lambda_n = (G^{n,1}_\lambda, G^{n,2}_\lambda, \ldots)$.
Kruskal from Erdős-Rényi

Consider the Erdős-Rényi random graph at the point $\lambda$ in the critical window: $G_{\lambda}^n = (G_{\lambda}^{n,1}, G_{\lambda}^{n,2}, \ldots)$.

Let $T_{\lambda}^n = (T_{\lambda}^{n,1}, T_{\lambda}^{n,2}, \ldots)$ be the components in the Kruskal process (which have the same vertex-sets but no cycles).
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Consider the Erdős-Rényi random graph at the point $\lambda$ in the critical window: $G^n_\lambda = (G^{n,1}_\lambda, G^{n,2}_\lambda, \ldots)$.

Let $T^n_\lambda = (T^{n,1}_\lambda, T^{n,2}_\lambda, \ldots)$ be the components in the Kruskal process (which have the same vertex-sets but no cycles).

For fixed $\lambda$ and $k$, we can construct $T^{n,k}_\lambda$ from $G^{n,k}_\lambda$ by running the cycle-breaking algorithm on the latter.
Kruskal from Erdős-Rényi

Reminder: cycle-breaking

► Start with all edges present. List them as $e_1, e_2, \ldots$ in decreasing order of weight.
► At step $i$, remove $e_i$ as long as it does not disconnect the graph (i.e. as long as it lies in a cycle).
► Stop when no cycles remain.
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Notice that only edges in cycles are affected by this procedure. So we could just ignore any edges which don’t lie in cycles.
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This tells us what the limiting analogue of the cycle-breaking procedure should be: repeatedly remove points chosen according to the uniform (Lebesgue) measure on the cycles in the metric space, until no cycles remain.
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This tells us what the limiting analogue of the cycle-breaking procedure should be: repeatedly remove points chosen according to the uniform (Lebesgue) measure on the cycles in the metric space, until no cycles remain.

(It’s actually a little delicate to check that the cycle-breaking procedure passes nicely to the limit . . . )
A limit for the Kruskal process

Let $T_\lambda^n = (T_\lambda^{n,1}, T_\lambda^{n,2}, \ldots)$ be the components of the Kruskal process, each endowed with the measure which assigns mass $n^{-2/3}$ to each vertex, and let $T_\lambda = (T_\lambda^1, T_\lambda^2, \ldots)$ be the sequence of spaces obtained from $G_\lambda$ by cycle-breaking.
A limit for the Kruskal process

Let $\mathbf{T}_\lambda^n = (T_{\lambda,1}^{n,1}, T_{\lambda,2}^{n,2}, \ldots)$ be the components of the Kruskal process, each endowed with the measure which assigns mass $n^{-2/3}$ to each vertex, and let $\mathcal{F}_\lambda = (\mathcal{F}_\lambda^1, \mathcal{F}_\lambda^2, \ldots)$ be the sequence of spaces obtained from $\mathcal{G}_\lambda$ by cycle-breaking.

**Theorem.** As $n \to \infty$,

$$\frac{1}{n^{1/3}} \mathbf{T}_\lambda^n \xrightarrow{d} \mathcal{F}_\lambda,$$

where convergence is with respect to the distance

$$\text{dist}_{\text{GHP}}^4(\mathcal{A}, \mathcal{B}) = \left( \sum_{i \geq 1} d_{\text{GHP}}(\mathcal{A}_i, \mathcal{B}_i)^4 \right)^{1/4}.$$
From Kruskal to the MST

- For fixed $n$, the process $(T_{\lambda}^{n,1}, \lambda \in \mathbb{R})$, which tracks the largest tree in the Kruskal forest, is eventually constant and equal to $M^n$, the MST of the complete graph.
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- For fixed $\lambda$, we have $n^{-1/3} T_{\lambda}^{n,1} \xrightarrow{d} \mathcal{T}_{\lambda}^{1}$ as $n \to \infty$ in $d_{GHP}$.

- The process $(T_{\lambda}^{1,1}, \lambda \in \mathbb{R})$ is eventually "increasing" in an appropriate sense.

- As $\lambda$ increases, we glue more and more little trees onto $T_{\lambda}^{1,1}$. However, its diameter remains bounded. Moreover, the additional mass coming from the little trees gets spread out essentially uniformly over the tree.

- The mass of $T_{\lambda}^{1,1}$ diverges as $\lambda \to \infty$, so we renormalise to get total mass measure 1 for each $\lambda$. If we do this, we obtain that there exists a limiting measured metric space $M$ as $\lambda \to \infty$. 
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The scaling limit of the MST

We obtain

$$\frac{1}{n^{1/3}} M^n \overset{d}{\to} M$$

as $n \to \infty$, in the sense of $d_{\text{GHP}}$. 
References

The continuum limit of critical random graphs
L. Addario-Berry, N. Broutin and C. Goldschmidt

Critical random graphs: limiting constructions and distributional properties
L. Addario-Berry, N. Broutin and C. Goldschmidt

The scaling limit of the minimum spanning tree of the complete graph
L. Addario-Berry, N. Broutin, C. Goldschmidt and G. Miermont