The impact of selection in the Λ-Wright-Fisher model

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Branching structures
The third Bath-Paris meeting
University of Bath, 9-11 June 2014

Wright-Fisher diffusion with selection

$$X_t = x + \sigma \int_{[0,t]} \sqrt{X_s(1-X_s)}dB_s - \alpha \int_0^t X_s(1-X_s)ds$$

- If $\alpha = 0$, this is the classical Wright-Fisher, which get absorbed in a finite time ζ in 0 or 1.
- If $\alpha > 0$, a selection term is added and X_t represents the frequency of the *disadvantaged* allele.

Standard methods for diffusion processes yield

$$\mathbb{P}_x[X_\infty = 0] = \frac{1-e^{-2x\alpha/\sigma^2}}{1-e^{-2\alpha/\sigma^2}} \in (0,1).$$

Λ-resampling

Let Λ be a finite measure on $]0,1]$, and $\alpha \geq 0$. Consider the following SDE

$$X_t = x + \int_{[0,t]\times[0,1]\times[0,1]} z\left(1_{u \leq X_{s-}} - X_{s-}\right)\bar{\mathcal{M}}(ds, du, dz) \\ - \alpha \int_0^t X_s(1 - X_s)ds$$

where $\mathcal{M}$ is a Poisson measure on $\mathbb{R}_+ \times [0,1] \times [0,1]$ with intensity

$$ds \otimes du \otimes z^{-2}\Lambda(dz).$$

The solution is a positive supermartingale and thus converges a.s to $X_\infty \in \{0,1\}$.

Interpretation of the SDE

- Denote the frequency of the allele just before time s by X_{s-}.

  If (s, u, z) is an atom of the measure $\mathcal{M}$, then, at time s,

  - if $u \leq X_{s-}$, the frequency of the allele increases by a fraction $z(1 - X_{s-})$
  - if $u > X_{s-}$, the frequency of the allele decreases by a fraction zX_{s-}.
- Continuously in time, the frequency decreases due to the deterministic selection mechanism.

Question: Are there Λ and α such that the disadvantaged allele becomes extinguished a.s?

Theorem (main result)

Let

$$\alpha^\star := -\int_0^1 \log(1-x)x^{-2}\Lambda(dx) \in (0,\infty].$$

Then,

1) *if* $\alpha < \alpha^\star$ *then* $\mathbb{P}_x[X_\infty = 0] \in]0,1[$, *for all* $x \in]0,1[$;
2) *if* $\alpha^\star < \infty$ *and* $\alpha > \alpha^\star$ *then* $X_\infty = 0$ *a.s.*

- Der, Epstein and Plotkin (*Genetics* 2012) consider $\Lambda = c\delta_x$.
- Bob Griffiths (2014) obtains the result for $\alpha^\star < \infty$ and shows that if $\alpha = \alpha^\star$ then $X_\infty = 0$ a.s.
- If $\int_0^1 x^{-1}\Lambda(dx) = \infty$ (dust-free) then $\alpha^\star = \infty$.
- Bah and Pardoux (2012) show that $(X_t, t \geq 0)$ is absorbed iff $\Lambda \in \mathbb{CDI}$.

Branching-coalescing dual process

Let $(R_t, t \geq 0)$ be the continuous-time Markov chain with values in $\mathbb{N} := \{1, 2, ...\}$ and generator:

$$\mathcal{L}g(n) := \underbrace{\sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}[g(n-k+1) - g(n)] + \alpha n[g(n+1) - g(n)]}_{\text{coagulations} \qquad \text{versus} \qquad \text{branching}}$$

with

$$\lambda_{n,k} := \int_0^1 x^k (1-x)^{n-k} x^{-2} \Lambda(dx).$$

Lemma (duality)

For all $x \in [0,1], n \geq 1,$

$$\mathbb{E}[X_t^n | X_0 = x] = \mathbb{E}[x^{R_t} | R_0 = n].$$

The asymptotics of $(X_t, t \geq 0)$ are related to those of $(R_t, t \geq 0)$

Lemma

1) *If* $(R_t, t \geq 0)$ *is positive recurrent then the law of* X_∞ *charges both* 0 *and* 1*.*
2) *If* $(R_t, t \geq 0)$ *is transient then* $X_\infty = 0$ *almost surely.*

Function δ

Define

$$\delta(n) := -n \int_0^1 \log\left(1 - \frac{1}{n}(nx - 1 + (1-x)^n)\right) x^{-2}\Lambda(dx)$$

Proposition (Möhle and Herriger 2013)

The maps δ and $n \mapsto \delta(n)/n$ are non-decreasing;

$$\delta(n)/n \underset{n\to\infty}{\longrightarrow} -\int_0^1 \log(1-x)x^{-2}\Lambda(dx) = \alpha^\star.$$

Case $\alpha < \alpha^\star$

Define

$$f(n) := \sum_{k=2}^{n} \frac{k}{\delta(k)} \log\left(\frac{k}{k-1}\right).$$

From Jensen inequality,

$$\frac{\delta(n)}{n} \leq \sum_{j=2}^{n} -\log\left(\frac{n-j+1}{n}\right)\binom{n}{j}\lambda_{n,j}$$

Plugging f in $\mathcal{L}$, one can show

$$\begin{aligned}\mathcal{L}f(n) &\leq -1 + \alpha\frac{n}{\delta(n)} \\ &\underbrace{\leq}_{n\geq n_0} -1 + \alpha\left(1/\alpha^\star + \epsilon\right) \\ &< 0 \qquad \text{for } \epsilon \text{ small enough and a certain } n_0.\end{aligned}$$

Set $T^{n_0} := \inf\{t; R_t < n_0\}$, we get

$$\begin{aligned}\mathbb{E}_n[f(R_{T^{n_0}})] &= f(n) + \mathbb{E}\left[\int_0^{T^{n_0}} \mathcal{L}f(R_s)ds\right] \\ &\leq f(n) + \left(-1 + \frac{\alpha}{\alpha^\star} + \epsilon\alpha\right)\mathbb{E}[T^{n_0}].\end{aligned}$$

Thus

$$\underbrace{(1 - \frac{\alpha}{\alpha^\star} - \epsilon\alpha)}_{>0}\mathbb{E}[T^{n_0}] \leq f(n) - \mathbb{E}_n[f(R_{T^{n_0}}] \leq f(n).$$

which establishes the positive recurrence.

Case $\alpha > \alpha^\star$

Let $g : n \mapsto 1/\log(n+1)$, one can show that there exists n_0 such that

$$\mathcal{L}g(n) < 0 \text{ for all } n \geq n_0.$$

Let $n > n_0$ and assume $\mathbb{P}_n[T^{n_0} < \infty] = 1$, then by the martingale convergence theorem

$$g(n) = \mathbb{E}[g(R_0)] \geq \mathbb{E}[g(R_{T^{n_0}})]$$

which is not possible since g is decreasing. Thus $\mathbb{P}_n[T^{n_0} < \infty] < 1$ and by irreducibility $R_t \longrightarrow \infty$ a.s.

Behind δ

Roughly speaking δ measures the coagulation strength in a logarithm scale. Indeed

$$\delta(n)/n = -\log\left(-\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\frac{N_t}{n}|N_0 = n\right]\right),$$

where $(N_t, t \geq 0)$ is the block-counting process of a Λ-coalescent. The logarithm scale allows us to compare the exponential growth due to the binary branching with the decay due to the coagulations.

Behind $\alpha^\star$

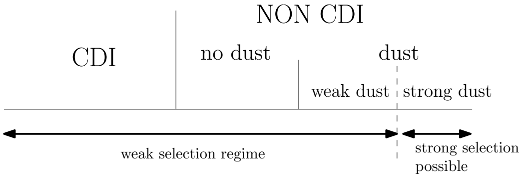
Let Π a Λ-coalescent, recall

$$\frac{\#\text{singletons in } \Pi_{|[n]}(t)}{n} \longrightarrow (\exp(-\xi_t), t \geq 0)$$

with $(\xi_t, t \geq 0)$ a drift-free subordinator with Laplace exponent

$$\Phi(q) := \int_0^1 (1 - (1-x)^q)\, x^{-2} \Lambda(dx).$$

One can easily check that $\alpha^\star = \mathbb{E}[\xi_1]$.



Adapting Haas and Miermont's work on scaling limits of **self-similar** Markov chains, if

$$h(u) = \int_{[u,1]} x^{-2}\Lambda(dx) \text{ is RV}(-\beta) \text{ with } \beta \in (0,1)$$

then $\left(\frac{1}{n}\mathrm{R}([h(1/n)t]), t \geq 0\right) \Longrightarrow \left(\exp\left(-(\xi_{C_t} - \alpha C_t)\right), t \geq 0\right)$ where

$$C_t := \inf\{u \geq 0 : \int_0^u \exp\left(-\beta(\xi_r - \alpha r)\right) dr > t\}$$

By classic results on exponential functional of Lévy process:

$$\int_0^\infty \exp\left(-\beta(\xi_r - \alpha r)\right) dr < \infty \text{ iff } \alpha < \alpha^\star.$$

Thank you

- Der, Epstein, Plotkin (2012) Dynamics of neutral and selected alleles when the offspring distribution is skewed. Genetics
- F. (2014): The impact of selection in the Λ-Wright-Fisher model. (ECP)
- Griffiths (2014), Lambda-Fleming-Viot process and a connection with Wright-Fisher diffusion.
- Möhle and Herriger (2013): conditions for exchangeable coalescent processes to come down from infinity (ALEA)
- Schweinsberg (2000) A necessary and sufficient condition for the Λ-coalescent to come down from infinity, (ECP)
- Schweinsberg (2000) Coalescents with simultaneous multiple collisions, EJP