MA40189 - Solution Sheet Zero

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1. **Whether certain mice are black or brown depends upon a pair of genes, each of which is either B or b. If both members of the pair are alike, the mouse is said to be homozygous, and if they are different it is said to be heterozygous. The mouse is brown only if it is homozygous bb. The offspring of a pair of mice have two such genes, one from each parent, and if the parent is heterozygous, the inherited gene is equally likely to be B or b. Suppose that a black mouse results from a mating between two heterozygotes.**

   (a) **What are the probabilities that this mouse is homozygous and that it is heterozygous?**

   As the two mated mice are heterozygotes then they are both Bb. Let G0 denote the event that the parents are both Bb and G1-BB, G1-Bb and G1-bb respectively denote the three events that the offspring is BB, Bb and bb. Let G1-black denote the event that the offspring is black and G1-brown that it is brown, so G1-black = G1-BB $\cup$ G1-Bb and G1-brown = G1-bb. As the inherited gene from a heterozygous parent is equally likely to be B or b we have

$$P(\text{G1-BB} \,|\, \text{G0}) \;=\; \frac{1}{4},\; P(\text{G1-Bb} \,|\, \text{G0}) \;=\; \frac{1}{2},\; P(\text{G1-bb} \,|\, \text{G0}) \;=\; \frac{1}{4}$$

   so that

$$\begin{aligned} P(\text{G1-black} \,|\, \text{G0}) &= P(\text{G1-BB} \,|\, \text{G0}) + P(\text{G1-Bb} \,|\, \text{G0}) \;=\; \frac{3}{4}, \\ P(\text{G1-brown} \,|\, \text{G0}) &= P(\text{G1-bb} \,|\, \text{G0}) \;=\; \frac{1}{4}. \end{aligned}$$

   Now, using conditional probability we have

$$\begin{aligned} P(\text{G1-BB} \,|\, \text{G1-black}, \text{G0}) &= \frac{P(\text{G1-BB}, \text{G1-black} \,|\, \text{G0})}{P(\text{G1-black} \,|\, \text{G0})} \\ &= \frac{P(\text{G1-BB} \,|\, \text{G0})}{P(\text{G1-black} \,|\, \text{G0})} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \;=\; \frac{1}{3}. \end{aligned}$$

   It follows immediately, as we know that the mouse is black, that

$$P(\text{G1-Bb} \,|\, \text{G1-black}, \text{G0}) \;=\; 1 - P(\text{G1-BB} \,|\, \text{G1-black}, \text{G0}) \;=\; \frac{2}{3}.$$

Letting G1-homozygous denote the event that the offspring is homozygous and G1-heterozygous that it is heterozygous we have

$$\begin{aligned} P(\text{G1-homozygous} \mid \text{G1-black}, \text{G0}) &= P(\text{G1-BB} \mid \text{G1-black}, \text{G0}) = \frac{1}{3}, \\ P(\text{G1-heterozygous} \mid \text{G1-black}, \text{G0}) &= P(\text{G1-Bb} \mid \text{G1-black}, \text{G0}) = \frac{2}{3}. \end{aligned}$$

Now suppose that this mouse is mated with a brown mouse, resulting in seven offspring, all of which turn out to be black.

(b) **Use Bayes' Theorem to find the probability that the black mouse was homozygous BB.**

Let G1 denote the event that the black mouse is mated with a brown mice and G2-7BO the event that the seven offspring of G1 are black. Then, assuming that each offspring is independent,

$$\begin{aligned} P(\text{G2-7BO} \mid \text{G1-BB}, \text{G1}, \text{G0}) &= 1, \\ P(\text{G2-7BO} \mid \text{G1-Bb}, \text{G1}, \text{G0}) &= \left(\frac{1}{2}\right)^7. \end{aligned}$$

These can be viewed as likelihoods. Using the theorem of total probability,

$$\begin{aligned} P(\text{G2-7BO} \mid \text{G1}, \text{G0}) &= P(\text{G2-7BO} \mid \text{G1-BB}, \text{G1}, \text{G0}) P(\text{G1-BB} \mid \text{G1}, \text{G0}) + \\ &\quad P(\text{G2-7BO} \mid \text{G1-Bb}, \text{G1}, \text{G0}) P(\text{G1-Bb} \mid \text{G1}, \text{G0}) \\ &= P(\text{G2-7BO} \mid \text{G1-BB}, \text{G1}, \text{G0}) P(\text{G1-BB} \mid \text{G1-black}, \text{G0}) + \\ &\quad P(\text{G2-7BO} \mid \text{G1-Bb}, \text{G1}, \text{G0}) P(\text{G1-Bb} \mid \text{G1-black}, \text{G0}) \\ &= 1\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)^7 \left(\frac{2}{3}\right) \\ &= \frac{1}{3}\left(1 + \frac{1}{2^6}\right). \end{aligned}$$

Thus, using Bayes' theorem we find

$$\begin{aligned} &P(\text{G1-BB} \mid \text{G2-7BO}, \text{G1}, \text{G0}) \\ &\quad = \frac{P(\text{G2-7B0} \mid \text{G1-BB}, \text{G1}, \text{G0}) P(\text{G1-BB} \mid \text{G1}, \text{G0})}{P(\text{G2-7BO} \mid \text{G1}, \text{G0})} \\ &\quad = \frac{1\left(\frac{1}{3}\right)}{\frac{1}{3}\left(1 + \frac{1}{2^6}\right)} \\ &\quad = \frac{2^6}{2^6 + 1} = \frac{64}{65}. \end{aligned}$$

Note the form of this equation. We can view $P(\text{G1-BB} \mid \text{G1}, \text{G0})$ as our prior probability for the event G1-BB, $P(\text{G2-7B0} \mid \text{G1-BB}, \text{G1}, \text{G0})$ our likelihood (the probability of observing the seven black offspring given that we knew G1-BB) and $P(\text{G1-BB} \mid \text{G2-7BO}, \text{G1}, \text{G0})$ our posterior probability for the event G1-BB. We shall make explicit use of this relationship in the next part of the question.

(c) **Recalculate the same probability by regarding the seven offspring as seven observations made sequentially, treating the posterior after each observation as the prior for the next.**

Let G2-BOi denote the event that the ith offspring is black and G2-iBO denote the event that there have been i black offspring. Thus, G2-iBO = G2-BOi $\cap$ G2-(i-1)BO. After observing $(i-1)$ offspring, the probability for the event G1-BB is $P(\text{G1-BB} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})$ whilst the probability for the event G1-Bb is $P(\text{G1-Bb} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})$. Given the ancestral history, we assume that the offspring are independent. In particular, for each $i = 1, \dots, 7$, we have the likelihoods

$$\begin{aligned} P(\text{G2-BOi} \mid \text{G2-(i-1)BO}, \text{G1-BB}, \text{G1}, \text{G0}) &= P(\text{G2-BOi} \mid \text{G1-BB}, \text{G1}, \text{G0}) = 1, \\ P(\text{G2-BOi} \mid \text{G2-(i-1)BO}, \text{G1-Bb}, \text{G1}, \text{G0}) &= P(\text{G2-BOi} \mid \text{G1-Bb}, \text{G1}, \text{G0}) = \frac{1}{2}. \end{aligned}$$

Then, using Bayes' theorem we have

$$\begin{aligned} &P(\text{G1-BB} \mid \text{G2-iBO}, \text{G1}, \text{G0}) \\ &\quad = P(\text{G1-BB} \mid \text{G2-B0i}, \text{G2-(i-1)BO}, \text{G1}, \text{G0}) \\ &\quad = \frac{P(\text{G2-B0i} \mid \text{G1-BB}, \text{G1}, \text{G0}) P(\text{G1-BB} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})}{P(\text{G2-BOi} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})}, \\ &P(\text{G1-Bb} \mid \text{G2-iBO}, \text{G1}, \text{G0}) \\ &\quad = P(\text{G1-Bb} \mid \text{G2-B0i}, \text{G2-(i-1)BO}, \text{G1}, \text{G0}) \\ &\quad = \frac{P(\text{G2-B0i} \mid \text{G1-Bb}, \text{G1}, \text{G0}) P(\text{G1-Bb} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})}{P(\text{G2-BOi} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})} \end{aligned}$$

Thus, noting that the denominator $P(\text{G2-BOi} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0})$ is common to both terms,

$$\begin{aligned} &P(\text{G1-BB} \mid \text{G2-iBO}, \text{G1}, \text{G0}) \\ &\quad \propto P(\text{G2-B0i} \mid \text{G1-BB}, \text{G1}, \text{G0}) P(\text{G1-BB} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0}), \\ &P(\text{G1-Bb} \mid \text{G2-iBO}, \text{G1}, \text{G0}) \\ &\quad \propto P(\text{G2-B0i} \mid \text{G1-Bb}, \text{G1}, \text{G0}) P(\text{G1-Bb} \mid \text{G2-(i-1)BO}, \text{G1}, \text{G0}) \end{aligned}$$

which are of the form "Posterior $\propto$ Likelihood $\times$ Prior". The constant of proportionality can be found by noting

$$P(\text{G1-BB} \mid \text{G2-iBO}, \text{G1}, \text{G0}) + P(\text{G1-Bb} \mid \text{G2-iBO}, \text{G1}, \text{G0}) = 1.$$

We now add the data about the seven offspring sequentially. Our initial priors are $P(\text{G1-BB} \mid \text{G1-black}, \text{G0}) = \frac{1}{3}$ for G1-BB and $P(\text{G1-Bb} \mid \text{G1-black}, \text{G0}) = \frac{2}{3}$ for G1-Bb. We now update these beliefs after observing the event G2-1BO. We have:

Event	Prior	Likelihood	Likelihood $\times$ Prior	Posterior
G1-BB	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$
G1-Bb	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$

We now add the second black offspring so that we want our beliefs after observing the event G2-2BO. We use our posteriors from the previous stage as our new priors, that is $P(\text{G1-BB}\,|\,\text{G2-1BO, G1, G0}) = \frac{1}{2}$ and $P(\text{G1-Bb}\,|\,\text{G2-1BO, G1, G0}) = \frac{1}{2}$ act as our priors. We have

Event	Prior	Likelihood	Likelihood × Prior	Posterior
G1-BB	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}} = \frac{2}{3}$
G1-Bb	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}} = \frac{1}{3}$

We can continue in this vein. A straightforward inductive argument shows that

$$\begin{aligned} P(\text{G1-BB}\,|\,\text{G2-iBO, G1, G0}) &= \frac{2^{i-1}}{1+2^{i-1}}, \\ P(\text{G1-BB}\,|\,\text{G2-iBO, G1, G0}) &= \frac{1}{1+2^{i-1}}. \end{aligned}$$

Adding the seventh black offspring thus gives

Event	Prior	Likelihood	Likelihood × Prior	Posterior
G1-BB	$\frac{32}{33}$	1	$\frac{32}{33}$	$\frac{\frac{32}{33}}{\frac{32}{33}+\frac{1}{66}} = \frac{64}{65}$
G1-Bb	$\frac{1}{33}$	$\frac{1}{2}$	$\frac{1}{66}$	$\frac{\frac{1}{66}}{\frac{32}{33}+\frac{1}{66}} = \frac{1}{65}$

We now adopt this explicit "Posterior ∝ Likelihood × Prior" approach for the answers to parts (a) and (b).

(a) Learning that G1-black.

Event	Prior	Likelihood	Likelihood × Prior	Posterior
G1-BB	$\frac{1}{4}$	1	$\frac{1}{4}$	$\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}+0} = \frac{1}{3}$
G1-Bb	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{\frac{1}{2}}{\frac{1}{4}+\frac{1}{2}+0} = \frac{2}{3}$
G1-bb	$\frac{1}{4}$	0	0	0

(b) Learning that G1-black mates with a brown mouse, resulting in seven black offspring. We can remove the event G1-bb as this has zero prior probability.

Event	Prior	Likelihood	Likelihood × Prior	Posterior
G1-BB	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}\left(\frac{1}{2}\right)^6} = \frac{64}{65}$
G1-Bb	$\frac{2}{3}$	$\left(\frac{1}{2}\right)^7$	$\frac{1}{3}\left(\frac{1}{2}\right)^6$	$\frac{\frac{1}{3}\left(\frac{1}{2}\right)^6}{\frac{1}{3}+\frac{1}{3}\left(\frac{1}{2}\right)^6} = \frac{1}{65}$

(d) **What is the probability that an eighth offspring is also black.**

Here we are looking at the predictive distribution. Let G2-BO8 denote the event that the eighth offspring is black. The solution is once again to use total probability with the two events G1-BB and G1-Bb. We have

$$\begin{aligned} & P(\text{G2-BO8}\,|\,\text{G2-7BO, G1, G0}) = \\ & \quad P(\text{G2-BO8}\,|\,\text{G2-7B0, G1-BB, G1, G0})P(\text{G1-BB}\,|\,\text{G2-7BO, G1, G0}) + \\ & \quad P(\text{G2-BO8}\,|\,\text{G2-7B0, G1-Bb, G1, G0})P(\text{G1-Bb}\,|\,\text{G2-7B0, G1, G0}) \\ & = 1\left(\frac{64}{65}\right) + \frac{1}{2}\left(\frac{1}{65}\right) = \frac{129}{130}. \end{aligned}$$