1. (a) State the definition of a conjugate family for a parameter θ with respect to a likelihood $f(x \mid \theta)$. [2]

(b) Suppose that the likelihood $f(x \mid \theta)$ belongs to the k-parameter regular exponential family. Show that a conjugate prior can be found in the $(k+1)$-parameter regular exponential family. [3]

Let $X_1, \ldots, X_n$ be exchangeable so that the X_i are conditionally independent given a parameter θ. Let $\theta = (\mu, \nu)$ and suppose that each $X_i \mid \theta \sim N(\mu, \nu)$.

(c) Show that

$$f(x \mid \theta) \;=\; (2\pi)^{-n/2}\nu^{-n/2}\exp\left\{-\frac{1}{2\nu}\left[(n-1)s^2 + n(\overline{x}-\mu)^2\right]\right\}$$

where $x = (x_1, \ldots, x_n)$ and $\overline{x}$, $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ are respectively the sample mean and variance. Hence, or otherwise, explain why $\overline{X}$ and S^2 are sufficient for $X = (X_1, \ldots, X_n)$ for learning about θ. [3]

(d) It is judged that the improper joint prior distribution $f(\mu, \nu) \propto \nu^{-1}$ is appropriate.

(i) By first deriving the posterior distribution $\mu, \nu \mid x$ where $x = (x_1, \ldots, x_n)$, or otherwise, show that $\nu \mid x \sim \text{Inv-gamma}((n-1)/2, (n-1)s^2/2)$. [4]

(ii) Let $Y = (n-1)s^2/\nu$. Using the result of part (d)(i), or otherwise, show that $Y \mid x \sim \chi^2_{n-1}$, the chi-squared distribution with $n-1$ degrees of freedom. Comment on why this result suggests that the prior $f(\mu, \nu) \propto \nu^{-1}$ may be viewed as noninformative.
(Hint: You may use, without proof, the property that if $A \sim \text{Gamma}(a, b)$ then $cA \sim \text{Gamma}(a, b/c)$ for any constant $c > 0$. You should clearly state any further properties of the Gamma distribution you use.) [4]

(e) Suppose now that the prior for θ is given hierarchically and it is judged that $\mu \mid \nu \sim N(\mu_0, \nu/\lambda)$, where μ_0 and λ are known constants, and $\nu \sim \text{Inv-gamma}(\alpha, \beta)$ for known constants α and β. Show that, with respect to the normal likelihood given in part (c), the prior distribution for $\theta = (\mu, \nu)$ is a conjugate prior.
(Hint: You may use, without proof, the result that for all μ,

$$a(\mu-b)^2 + c(\mu-d)^2 \;=\; (a+c)\left(\mu - \frac{ab+cd}{a+c}\right)^2 + \left(\frac{ac}{a+c}\right)(b-d)^2$$

for any $a, b, c, d \in \mathbb{R}$ with $a \neq -c$.) [4]

2. (a) Let $X_1, \ldots, X_n$ be exchangeable so that the X_i are conditionally independent given a parameter θ. Suppose that $X_i \,|\, \theta \sim \text{Exp}(\theta)$ so that $E(X_i \,|\, \theta) = 1/\theta$.

(i) Find the Jeffreys prior for θ and show that the posterior distribution for θ given $x = (x_1, \ldots, x_n)$ is Gamma(α_n, β_n), stating the values of α_n and β_n. [4]

(ii) By considering the posterior mean, or otherwise, briefly explain how, in this case, the Jeffreys prior can be viewed as noninformative. [2]

(iii) Consider a further observation Z which is exchangeable with $X = (X_1, \ldots, X_n)$ where $n > 2$. Without calculating the predictive distribution of Z given $X = x$, find $E(Z \,|\, X)$ and $Var(Z \,|\, X)$. [3]

(iv) Suppose now that the prior for θ is instead given by the probability density function

$$f(\theta) \quad = \quad \frac{\beta^{\alpha}}{4\Gamma(\alpha)}\theta^{\alpha-1}\exp\left(-\beta\theta\right) + \frac{3\beta^{\alpha+1}}{4\Gamma(\alpha+1)}\theta^{\alpha}\exp\left(-\beta\theta\right).$$

for known constants $\alpha, \beta > 0$. Show that the posterior probability density function can be written as

$$f(\theta \,|\, x) \quad = \quad c f_1(\theta) + (1-c) f_2(\theta)$$

where

$$c \quad = \quad \frac{\alpha\left(\beta + \sum_{i=1}^{n} x_i\right)}{\alpha\left(\beta + \sum_{i=1}^{n} x_i\right) + 3\beta\left(\alpha + n\right)}$$

and $f_1(\theta)$ and $f_2(\theta)$ are probability density functions. [5]

(b) (i) State the definition of finite exchangeability. [2]

(ii) Let $X_1, \ldots, X_n$ be conditionally independent given a parameter $\theta = (\theta_1, \ldots, \theta_n)$ and suppose that each $X_i \,|\, \theta$ is from the same family of distributions with likelihood $f(x_i \,|\, \theta) = f(x_i \,|\, \theta_i)$. It is judged that $\theta_1, \ldots, \theta_n$ are independent and identically distributed with each θ_i having probability density function $\pi(\theta_i)$. Show that $X_1, \ldots, X_n$ are finitely exchangeable. [4]

3. Let $X_1, \ldots, X_n$ be conditionally independent given a parameter $\theta = (\theta_1, \ldots, \theta_n)$ and suppose that $X_i \,|\, \theta \sim \text{Po}(s_i\theta_i)$ where each s_i is known. It is judged that $\theta_1, \ldots, \theta_n$ are conditionally independent given ϕ with $\theta_i \,|\, \phi \sim \text{Exp}(\phi)$, so that $E(\theta_i \,|\, \phi) = 1/\phi$, and $\phi \sim \text{Gamma}(\alpha, \beta)$, where α and β are known.

(a) Show that the posterior density $f(\theta_1, \ldots, \theta_n, \phi \,|\, x)$, where $x = (x_1, \ldots, x_n)$, can be expressed as

$$f(\theta_1, \ldots, \theta_n, \phi \,|\, x) \quad \propto \quad \left(\prod_{i=1}^{n} \theta_i^{x_i}\right) \phi^{\alpha+n-1} \exp\left\{-\phi\left(\beta + \sum_{i=1}^{n} \theta_i\right) - \sum_{i=1}^{n} s_i\theta_i\right\}.$$

[2]

(b) Describe how to use the Gibbs sampler to sample from the posterior distribution of $\theta_1, \ldots, \theta_n, \phi \,|\, x$, deriving any required conditional distributions. [7]

Consider a general problem in which we will observe data X, where the distribution of X depends upon an unknown parameter θ, and we wish to make inference about θ.

(c) We wish to use the Metropolis-Hastings algorithm to sample from the posterior distribution of $\theta \,|\, x$ with density $\pi(\theta) = f(\theta \,|\, x)$. Letting $q(\phi \,|\, \theta)$ denote the proposal distribution when the current state is θ and $\alpha(\theta, \phi)$ the probability of accepting a move from θ to ϕ, describe how the algorithm works in this case. [4]

(d) Suppose that you can sample from a distribution $q(\theta)$ which is an approximation to the posterior distribution $f(\theta \,|\, x)$. Explain how to use the method of importance sampling to estimate $E(g(\theta) \,|\, X)$ for some function $g(\theta)$. [3]

(e) Let $\hat{I}$ denote the importance sampling estimator of $E(g(\theta) \,|\, X)$ constructed in part (d). By considering the variance of $\hat{I}$ with respect to the distribution $q(\theta)$, or otherwise, explain why a sensible choice of $q(\theta)$ is that which minimises $E\left(\frac{g^2(\theta)f(\theta \,|\, x)}{q(\theta)} \,\middle|\, X\right)$. [4]

4. Consider a statistical decision problem $[\Theta, \mathcal{D}, \pi(\theta), L(\theta, d)]$ for a univariate parameter θ and loss function

$$L(\theta, d) \quad = \quad g(\theta)(\theta - d)^2$$

where $g(\theta) > 0$ and d is a point estimate of θ.

(a) Show that the Bayes decision rule against $\pi(\theta)$ is

$$d^* \quad = \quad \frac{E(g(\theta)\theta)}{E(g(\theta))}$$

with corresponding Bayes risk

$$\rho^*(\pi) \quad = \quad E(g(\theta)\theta^2) - \frac{\{E(g(\theta)\theta)\}^2}{E(g(\theta))},$$

interpreting the results for the case when $g(\theta) = 1$. [4]

Let $X_1, \ldots, X_n$ be exchangeable so that the X_i are conditionally independent given a parameter θ. Suppose that $X_i \,|\, \theta$ is distributed as a Weibull distribution with known shape parameter $\lambda > 0$ and unknown scale parameter $\theta > 0$, denoted $\mathrm{WE}(\lambda, \theta)$, with probability density function

$$f(x_i \,|\, \theta) \quad = \quad \begin{cases} \theta\lambda x_i^{\lambda-1}\exp\left(-\theta x_i^{\lambda}\right) & 0 < x_i < \infty \\ 0 & \text{otherwise,} \end{cases}$$

and $\theta \sim \mathrm{Gamma}(\alpha, \beta)$ where $\alpha > 3$ and β are known. We wish to produce a point estimate d for θ, with loss function

$$L(\theta, d) \quad = \quad \frac{(\theta - d)^2}{\theta^3}.$$

(b) Find the Bayes rule and Bayes risk of an immediate decision. [5]

(c) Find the Bayes rule and Bayes risk after observing $x = (x_1, \ldots, x_n)$. [3]

(d) Suppose that $\lambda = 1$ so that $X_i \,|\, \theta \sim \mathrm{Exp}(\theta)$. If each observation costs a fixed amount c then R_n, the total risk of a sample of size n, is the sum of the sample cost and the Bayes risk of the sampling procedure. Find R_n and thus the optimal choice of n. [4]

Let $\theta \in \mathbb{R}$ be a continuous univariate random variable with finite expectation. We wish to construct an interval estimate $d = (\theta_1, \theta_2)$, where $\theta_1 \leq \theta_2$, for θ. Let $\mathcal{D}_1$ denote the set of all such intervals d. Consider the loss function

$$L_1(\theta, d) \quad = \quad \theta_2 - \theta_1 + \begin{cases} \frac{2}{\alpha}(\theta_1 - \theta) & \text{if } \theta < \theta_1 \\ 0 & \text{if } \theta_1 \leq \theta < \theta_2 \\ \frac{2}{\alpha}(\theta - \theta_2) & \text{if } \theta_2 < \theta. \end{cases}$$

(e) By considering derivatives of the expected loss with respect to θ_1 and θ_2, or otherwise, show that, for the decision problem $[\Theta, \mathcal{D}_1, \pi(\theta), L_1(\theta, d)]$, the Bayes rule is $d^* = (\theta_1^*, \theta_2^*)$ where $P(\theta \leq \theta_1^*) = \alpha/2 = P(\theta \geq \theta_2^*)$. [4]