24/03/2022. PROBLEMS CLASS SIX – NOTES MADE ON THE VISUALISER.

Question Sheet Four.

2. $X_i \mid \theta \sim \text{Geom}(\theta)$ $\quad f(x_i \mid \theta) = (1-\theta)^{x_i - 1}\,\theta \quad x_i = 1, 2, \ldots$

$X = (X_1, \ldots, X_n)$ $\quad f(x \mid \theta) = (1-\theta)^{\sum_{i=1}^n x_i - n}\,\theta^n$

Conjugate prior $\quad \theta \sim \text{Beta}(\alpha, \beta)$, ~~$\theta \mid x \sim \text{Beta}(\alpha+1, \beta$~~

$\theta \mid x \sim \text{Beta}\left(\alpha + n, \beta + \sum_{i=1}^n x_i - n\right)$

$\alpha \mapsto \alpha + n, \quad \beta \mapsto \beta + \sum_{i=1}^n x_i - n$

Could relabel $\alpha_n = \alpha + n, \quad \beta_n = \beta + \sum_{i=1}^n x_i - n.$

(i) Consider the prior $\quad f(\theta) = \dfrac{1}{2\beta(\alpha+1, \beta)}\,\theta^{\alpha}(1-\theta)^{\beta-1} + \dfrac{1}{2\beta(\alpha, \beta+1)}\,\theta^{\alpha-1}(1-\theta)^{\beta}$

$$= \frac{1}{2}\tilde{f}_1(\theta) + \frac{1}{2}\tilde{f}_2(\theta)$$

where $\tilde{f}_1(\theta)$ is a pdf of $\text{Beta}(\alpha+1, \beta)$, $\tilde{f}_2(\theta)$ is pdf of $\text{Beta}(\alpha, \beta+1)$

The prior is a MIXTURE of two members of the conjugate family.

$$f(\theta \mid x) \propto f(x \mid \theta) f(\theta)$$

$$= \frac{1}{2}\underbrace{\tilde{f}_1(\theta) f(x \mid \theta)}_{\text{kernel of } \text{Beta}(\alpha_n+1, \beta_n)} + \frac{1}{2}\underbrace{\tilde{f}_2(\theta) f(x \mid \theta)}_{\text{kernel of } \text{Beta}(\alpha_n, \beta_n+1)}$$

Note: $\beta(\alpha+1, \beta) = \dfrac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \alpha\left(\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\right)$

$\beta(\alpha, \beta+1) = \dfrac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} = \beta\left(\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\right)$

Thus, $f(\theta) \propto \frac{1}{\alpha}\theta^{\alpha}(1-\theta)^{\beta-1} + \frac{1}{\beta}\theta^{\alpha-1}(1-\theta)^{\beta}$ $(\times \alpha\beta)$

wrt θ

$$\propto \beta\,\theta^{\alpha}(1-\theta)^{\beta-1} + \alpha\,\theta^{\alpha-1}(1-\theta)^{\beta}$$

$$f(\theta|x) \propto \theta^{n}(1-\theta)^{\sum_{i=1}^{n}x_i - n}\left[\beta\,\theta^{\alpha}(1-\theta)^{\beta-1} + \alpha\,\theta^{\alpha-1}(1-\theta)^{\beta}\right]$$

$$= \beta\,\theta^{\alpha+n}(1-\theta)^{\beta+\sum_{i=1}^{n}x_i-n-1} + \alpha\,\theta^{\alpha+n-1}(1-\theta)^{\beta+\sum_{i=1}^{n}x_i-n}$$

$$= \beta\,\underbrace{\theta^{\alpha_n}(1-\theta)^{\beta_n-1}}_{\text{kernel of Beta}(\alpha_n+1,\,\beta_n)} + \alpha\,\underbrace{\theta^{\alpha_n-1}(1-\theta)^{\beta_n}}_{\text{kernel of Beta}(\alpha_n,\,\beta_n+1)}$$

Let $f_1(\theta)$ be the pdf of Beta(α_n+1, β_n) and $f_2(\theta)$ be the pdf of Beta(α_n, β_n+1)

$$f(\theta|x) \propto \beta\,B(\alpha_n+1, \beta_n)f_1(\theta) + \alpha\,B(\alpha_n, \beta_n+1)f_2(\theta)$$

$$\propto \beta\alpha_n f_1(\theta) + \alpha\beta_n f_2(\theta)$$

As $\int_0^1 f(\theta|x)\,d\theta = 1 = \int_0^1 f_1(\theta)\,d\theta = \int_0^1 f_2(\theta)\,d\theta$ then:

$$f(\theta|x) = \frac{\beta\alpha_n f_1(\theta) + \alpha\beta_n f_2(\theta)}{\beta\alpha_n + \alpha\beta_n}$$

$$= \lambda f_1(\theta) + (1-\lambda)f_2(\theta)$$

where $\lambda = \frac{\beta\alpha_n}{\beta\alpha_n + \alpha\beta_n} = \frac{(\alpha+n)\beta}{(\alpha+n)\beta + \alpha(\beta + \sum_{i=1}^{n}x_i - n)}$

QUESTION SHEET FIVE.

4. Jeffreys prior for the binomial and the negative-binomial.

a). $f_X(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \qquad x = 0, 1, \ldots, n.$

$$\log f_X(x|\theta) = \log \binom{n}{x} + x \log \theta + (n-x) \log (1-\theta)$$

$$\frac{d}{d\theta} \log f_X(x|\theta) = \frac{x}{\theta} - \frac{(n-x)}{1-\theta}$$

$$\frac{d^2}{d\theta^2} \log f_X(x|\theta) = \frac{-x}{\theta^2} - \frac{(n-x)}{(1-\theta)^2}$$

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} \log f_X(x|\theta) \middle| \theta \right] = \frac{E(X|\theta)}{\theta^2} + \frac{n - E(X|\theta)}{(1-\theta)^2}$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta} = \frac{n}{\theta(1-\theta)}$$

The Jeffreys prior is $f(\theta) \propto \sqrt{I(\theta)} \propto \theta^{-1/2} (1-\theta)^{-1/2}$

This is Beta $(1/2, 1/2)$, a proper distribution.

$f_Y(y|\theta) = \binom{y-1}{r-1} \theta^r (1-\theta)^{y-r} \qquad y = r, r+1, \ldots$

$$\frac{d^2}{d\theta^2} \log f_Y(y|\theta) = \frac{-r}{\theta^2} - \frac{(y-r)}{(1-\theta)^2}$$

Then, $I(\theta) = \dfrac{r}{\theta^2} + \dfrac{(E(Y|\theta) - r)}{(1-\theta)^2} \qquad E(Y|\theta) = \dfrac{r}{\theta}$

$$= \frac{r}{\theta^2} + \frac{r}{\theta(1-\theta)}$$

$$= \frac{r}{\theta^2(1-\theta)}$$

The Jeffreys prior is $f(\theta) \propto \sqrt{I(\theta)} \propto \theta^{-1} (1-\theta)^{-1/2}$

This is the improper "Beta $(0, 1/2)$"

The Jeffreys prior for the binomial and negative-binomial differ.

Jeffreys Prior	Likelihood	Posterior
$\theta \sim \text{Beta}(1/2, 1/2)$	$X \mid \theta \sim \text{Bin}(n, \theta)$	$\theta \mid x \sim \text{Beta}(1/2 + x, 1/2 + n - x)$
$\theta \sim \text{Beta}(0, 1/2)$	$Y \mid \theta \sim \text{NBin}(r, \theta)$	$\theta \mid x \sim (r, 1/2 + y - r)$

If $r = x$, $y = n$ then in both cases I see n trials, x successes.

Thus, $f_X(x \mid \theta) \propto \theta^x (1-\theta)^{n-x}$

$f_Y(y \mid \theta) \propto \theta^x (1-\theta)^{n-x}$

However, the posteriors differ: $\theta \mid x \sim \text{Beta}(1/2 + x, 1/2 + n - x)$

$\theta \mid y \sim \text{Beta}(x, 1/2 + n - x)$

The Jeffreys prior depends on the form of the data. This is a violation of the LIKELIHOOD PRINCIPLE.