10/03/2022. PROBLEMS CLASS 4: NOTES MADE ON THE VISUALISER.

Conjugate Priors.

1 PARAMETER		2 PARAMETER
$X \mid \theta \sim Bin(n, \theta)$		$\theta \sim Beta(\alpha, \beta)$
$X \mid \theta \sim N(\theta, \sigma^2)$ (σ^2 known)	$\xrightarrow[\text{STATISTICS}]{\text{SUFFICIENT}}$	$\theta \sim N(\mu_0, \sigma_0^2)$
$X \mid \theta \sim N(\mu, \theta)$ (μ known)		$\theta \sim \text{Inv-Gamma}(\alpha_0, \beta_0)$

k-parameter exponential family.

$$f(x \mid \theta) = \exp\left\{ \sum_{j=1}^{k} \phi_j(\theta) u_j(x) + g(\theta) + h(x) \right\}$$

(k+1) functions of θ.

3. $$f(x_i \mid \theta) = \frac{1}{2\theta} \exp\left\{ -\frac{|x_i|}{\theta} \right\} \quad -\infty < x_i < \infty$$

[As a function of θ, looks like Inv-gamma$(0, |x_i|)$, kernel of Inv-gamma(α, β) is $\theta^{-(\alpha+1)} \exp\left\{ -\frac{\beta}{\theta} \right\}$]

$$= \exp\left\{ -\frac{|x_i|}{\theta} - \log\theta - \log 2 \right\}$$

$$\phi_1(\theta) = -\frac{1}{\theta}, \quad u_1(x_i) = |x_i|, \quad g(\theta) = -\log\theta, \quad h(x) = -\log 2$$

Thus, the Double-exponential is a member of the 1-parameter exponential family over x_i.

Consider that we wish to observe $x = (x_1, \dots, x_n)$ where each $X_i \mid \theta$ is the double exponential.

$$f(x|\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} \exp\left\{ -\frac{|x_i|}{\theta} \right\}$$

$$= 2^{-n} \theta^{-n} \exp\left\{ -\frac{1}{\theta} \sum_{i=1}^{n} |x_i| \right\}$$

This also looks like a kernel of Inv-gamma, in this case: Inv-gamma$\left(n-1, \sum_{i=1}^{n} |x_i|\right)$ as a function of θ.

$$= \exp\left\{ -\frac{1}{\theta} \sum_{i=1}^{n} |x_i| - n \log \theta - n \log 2 \right\}$$

$$= g(t, \theta) \tilde{h}(x) \quad \text{where } \tilde{h}(x) = 2^{-n}.$$

Sufficiency st $t(x) = \left(n, \sum_{i=1}^{n} |x_i|\right)$ (two statistics).

$$= \left(n, \sum_{i=1}^{n} u_1(x_i)\right)$$

As $f(x|\theta)$ looks like a kernel of Inv-gamma, take a generic Inv-gamma as a prior

$$f(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{-(\alpha+1)} \exp\left\{ -\frac{\beta}{\theta} \right\}$$

$$= \exp\left\{ -\frac{\beta}{\theta} - (\alpha+1) \log \theta + \alpha \log \beta - \log \Gamma(\alpha) \right\}$$

$$= \exp\left\{ a_1(\underbrace{\alpha, \beta}_{\text{hyperparameters}}) \phi_1(\theta) + d(\underbrace{\alpha, \beta}_{\text{hyperparameters}}) g(\theta) + c(a, d) \right\}$$

Member of a two-parameter exponential family over θ.

[$f(\theta) \propto \exp\left\{ a_1(\alpha, \beta) \phi_1(\theta) + d(\alpha, \beta) g(\theta) \right\}$

$$f(\theta|x) \propto f(x|\theta) f(\theta)$$

As $f(x|\theta) = g(t,\theta)\hat{h}(x)$ then:

$$f(\theta|x) \propto g(t,\theta) f(\theta)$$

$$= \theta^{-n} \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|\right\} \theta^{-(\alpha+1)} \exp\left\{-\frac{\beta}{\theta}\right\}$$

$$= \theta^{-(\alpha+n+1)} \exp\left\{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n}|x_i|\right)\right\}$$

Recognise this as a kernel of Inv-gamma$(\alpha+n, \beta + \sum_{i=1}^{n}|x_i|)$

i.e. $\theta|x \sim$ Inv-gamma$(\alpha+n, \beta + \sum_{i=1}^{n}|x_i|)$

Prior and posterior are in the same family giving conjugacy.

4. $f(x_1,\dots,x_n) = n!\left(1+\sum_{i=1}^{n}x_i\right)^{-(n+1)}$

Show they can be represented as IID exponentials.

If $X_i|\theta \sim \text{Exp}(\theta)$ then $\prod_{i=1}^{n} f(x_i|\theta) = \theta^n \exp\left(-\theta\sum_{i=1}^{n}x_i\right)$

$$\Rightarrow \underset{\Gamma(n+1)}{n!}\left(1+\sum_{i=1}^{n}x_i\right)^{-(n+1)} = \int_\theta \left\{\prod_{i=1}^{n} f(x_i|\theta)\right\} f(\theta)\, d\theta$$

$$= \int_\theta \theta^n \exp\left(-\theta\sum_{i=1}^{n}x_i\right) f(\theta)\, d\theta$$

$$= \int_\theta \theta^{(n+1)-1} \exp\left(-\theta\sum_{i=1}^{n}x_i\right) f(\theta)\, d\theta$$

Take $f(\theta) = \exp(-\theta)$ [Gamma(1,1)]

$$= \int_\theta \underbrace{\theta^{(n+1)-1} \exp\left(-\theta\left(1+\sum_{i=1}^{n}x_i\right)\right)}_{\text{kernel of Gamma}(n+1,\ 1+\sum_{i=1}^{n}x_i)} d\theta$$