03/03/2022. PROBLEMS CLASS 3 – NOTES MADE ON THE VISUALISER.

2. Idea behind this question is that if $X_1, \dots, X_n$ are finitely exchangeable then the joint distribution of any subcollection of the variables does not depend upon the variables chosen.

e.g. Marginal distribution of each X_i is the same

Pairwise joint distribution of (X_i, X_j) does not depend on $i \neq j$

[e.g. X_1, X_2, X_3 discrete exchangeable RVs,

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \mathbb{P}(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, X_3 = x_{\pi(3)})$$

i.e. there's an i, j such that $x_i = x_{\pi(j)}$

$$\mathbb{P}(X_1 = x_1) = \sum_{x_2} \sum_{x_3} \mathbb{P}(X_1 = x_1, X_2 = x_2, X_3 = x_3)$$

$$= \sum_{x_2} \sum_{x_3} \mathbb{P}(X_1 = x_2, X_2 = x_1, X_3 = x_3)$$

$= \mathbb{P}(X_2 = x_1)$ so X_1, X_2 have the same distribution]

3. $X \mid \mu \sim N(\mu, \sigma^2)$ $\quad Y \mid \mu, \delta \sim N(\mu + \delta, \sigma^2)$ $\quad \sigma^2$ known.

Assume $X \perp\!\!\!\perp Y \mid \mu, \delta$.

a). Calculate the joint distribution of X, Y given μ, δ.

$$f(x, y \mid \mu, \delta) = f(x \mid \mu, \delta) f(y \mid \mu, \delta)$$

$$= f(x \mid \mu) f(y \mid \mu, \delta)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y-(\mu+\delta))^2\right\}$$

$$= \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 + (y-(\mu+\delta))^2\right]\right\}$$

b. $f(\mu,\delta) \propto 1$ "improper prior" as it doesn't integrate to a pdf.

[Motivation: $X|\mu \sim N(\mu,\sigma^2)$ σ^2 known. The conjugate prior is $\mu \sim N(\mu_0,\sigma_0^2)$ so $\mu|x \sim N(\mu_1,\sigma_1^2)$

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \; ; \; \mu_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{x}{\sigma^2}\right)$$

As σ_0^2 increases, we have a weaker and weaker prior. As $\sigma_0^2 \to \infty$ then $\sigma_1^2 \to \sigma^2$ and $\mu \to x$ and the posterior is informed only by the data].

$$f(\mu,\delta|x,y) \propto f(x,y|\mu,\delta)f(\mu,\delta)$$
$$\propto f(x,y|\mu,\delta)$$

Posterior is the standardised likelihood.

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left[\mu^2 - 2x\mu + (\mu+\delta)^2 - 2y(\mu+\delta)\right]\right\}$$
$$= \exp\left\{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2(x+y)\mu + 2\mu\delta + \delta^2 - 2y\delta\right]\right\}$$

Note: $\mu\delta$ term means that $\mu|x,y$ and $\delta|x,y$ not independent.

c). $$f(\delta|x,y) = \int_{-\infty}^{\infty} f(\mu,\delta|x,y)\,d\mu$$

$$\propto \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2(x+y)\mu + 2\mu\delta + \delta^2 - 2y\delta\right]\right\} d\mu$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[\delta^2 - 2y\delta\right]\right\} \int_{-\infty}^{\infty} \underbrace{\exp\left\{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2(x+y-\delta)\mu\right]\right\}}_{\text{kernel of } N\left(\frac{x+y-\delta}{2},\ \frac{\sigma^2}{2}\right)} d\mu$$

Missing terms from the kernel involve δ so we have to account for these.

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2(x+y-\delta)\mu\right]\right\} d\mu$$

$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{2}{\sigma^2}\right)\left(\mu - \frac{x+y-\delta}{2}\right)^2\right\} \exp\left\{-\frac{1}{\sigma^2}\left(-\frac{(x+y-\delta)^2}{4}\right)\right\} d\mu$$

$$\propto \exp\left\{-\frac{1}{\sigma^2}\left(-\frac{(x+y-\delta)^2}{4}\right)\right\}$$

$$\propto \exp\left\{\frac{1}{\sigma^2}\,\frac{\left[\delta^2 - 2(x+y)\delta\right]}{4}\right\}$$

Thus,

$$f(\delta|x,y) \propto \exp\left\{-\frac{1}{2\sigma^2}\left[\delta^2 - 2y\delta\right]\right\} \exp\left\{+\frac{1}{\sigma^2}\,\frac{\left[\delta^2 - 2(x+y)\delta\right]}{4}\right\}$$

$$= \exp\left\{\frac{-1}{2(2\sigma^2)}\left[\delta^2 - 2(y-x)\delta\right]\right\}$$

i.e. $\delta|x,y \sim N(y-x, 2\sigma^2)$.