17/2/2022. PROBLEMS CLASS 1: NOTES MADE ON THE VISUALISER

For a RV X with density $f(x)$ if

proportionality wrt rv

$$f(x) = c\,q(x) \quad \text{i.e. } f(x) \propto q(x)$$

where c is a constant not depending upon x then $q(x)$ is a KERNEL of the density.

e.g. $\theta \sim \text{Beta}(\alpha, \beta)$

$$f(\theta) = \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$\propto \underbrace{\theta^{\alpha-1}(1-\theta)^{\beta-1}}_{\text{A kernel of Beta}(\alpha,\beta).}$$

((Compare Q1)

Suppose $X \mid \theta \sim \text{Bin}(n, \theta)$

$$f(x \mid \theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$$

wrt x

A kernel of this is $f(x \mid \theta) \propto \dfrac{1}{x!\,(n-x)!} \theta^{x} (1-\theta)^{-x}$

Viewing this as a likelihood function $L(\theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$

as a function of θ

$$\propto \underbrace{\theta^{x}(1-\theta)^{n-x}}_{\text{looks like the kernel of a Beta distribution as a function of } \theta.}$$

4. $\theta \sim \text{Beta}(4,4)$ $\quad X|\theta \sim \text{Bin}(10, \theta)$. Observe $X < 3$.

$$f(\theta|x) \propto \mathbb{P}(X < 3 \mid \theta) f(\theta)$$

wrt θ

[If I have a kernel $q(\theta)$ for $f(\theta)$ i.e. $f(\theta) = c\,q(\theta)$ then

$$f(\theta|x) \propto \mathbb{P}(\cancel{X<3}\ f(x|\theta)\, q(\theta)$$

and if the likelihood, as a function of θ, $L(\theta) = d\, r(\theta)$ then

$$f(\theta|x) \propto \underbrace{r(\theta)}_{\text{Not a kernel of } f(x|\theta) \text{ as a function of } x.} q(\theta) \quad]$$

$$= \left\{ \sum_{x=0}^{2} \binom{10}{x} \theta^{x} (1-\theta)^{n-x} \right\} \frac{1}{\beta(4,4)} \theta^{4-1} (1-\theta)^{4-1}$$

$$\propto \sum_{x=0}^{2} \binom{10}{x} \underbrace{\theta^{(4+x)-1} (1-\theta)^{(4+n-x)-1}}_{\text{kernel of } \text{Beta}(4+x,\ 4+n-x)}$$

[Compare with Q2]

Hence, $f(\theta|x) = k \left[\sum_{x=0}^{2} \binom{10}{x} \theta^{(4+x)-1} (1-\theta)^{(4+n-x)-1} \right]$

[NB. $n = 10$]

As $\int_0^1 f(\theta|x)\, d\theta = 1$ then

$$k = \left[\int_0^1 \sum_{x=0}^{2} \binom{10}{x} \theta^{(4+x)-1} (1-\theta)^{(14-x)-1} d\theta \right]^{-1}$$

$$= \left[\sum_{x=0}^{2} \binom{10}{x} \int_0^1 \theta^{(4+x)-1} (1-\theta)^{(14-x)-1} d\theta \right]^{-1}$$

Recognising kernels can help solve sum (discrete RVs) and integrals (continuous RVs) [Compare with Q3]

$$k = \left[\sum_{x=0}^{2} \binom{10}{x} \beta(4+x, 14-x) \right]^{-1}$$

Thus, $$k^{-1} = \sum_{x=0}^{2} \binom{10}{x} \beta(4+x, 14-x)$$

$$= \beta(4,14) + 10\,\beta(5,13) + 45\,\beta(6,12)$$

$$= \frac{\Gamma(4)\Gamma(14)}{\Gamma(18)} + 10\,\frac{\Gamma(5)\Gamma(13)}{\Gamma(18)} + 45\,\frac{\Gamma(6)\Gamma(12)}{\Gamma(18)}$$

$$= \frac{\Gamma(4)\Gamma(12)}{\Gamma(18)} \left[\overset{156}{(13 \times 12)} + \overset{480}{10(4 \times 12)} + \overset{900}{45(5 \times 4)} \right]$$

$$= 1536\,\frac{\Gamma(4)\Gamma(12)}{\Gamma(18)}$$

i.e. $$f(\theta|x) = \frac{\Gamma(18)}{1536\,\Gamma(4)\Gamma(12)} \left[\sum_{x=0}^{2} \binom{10}{x} \theta^{(4+x)-1} (1-\theta)^{(14-x)-1} \right]$$

This is a MIXTURE DISTRIBUTION.

$$f(\theta|x) = \alpha_0 f_0(\theta) + \alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$$

where $f_i(\theta)$ is the pdf of $\text{Beta}(4+i, 14-i)$, $\alpha_i \geq 0$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$. [Here $\alpha_0 = \frac{156}{1536}$; $\alpha_1 = \frac{480}{1536}$; $\alpha_2 = \frac{900}{1536}$]

This is a weighted sum.