08/03/2022. LECTURE 9 - NOTES MADE ON THE VISUALISER.

DEFINITION (INFINITE EXCHANGEABILITY)

The infinite sequence of RVs $X_1, X_2, \ldots$ are judged to be INFINITELY EXCHANGEABLE if every finite subsequence is judged (finitely) exchangeable.

A natural question of interest is whether every finitely exchangeable sequence can be embedded into an infinitely exchangeable sequence?

EXAMPLE.

Suppose that X_1, X_2, X_3 are three exchangeable events (i.e. 0-1 RVs) with

$$\mathbb{P}(X_1=0, X_2=1, X_3=1) = \mathbb{P}(X_1=1, X_2=0, X_3=1)$$
$$= \mathbb{P}(X_1=1, X_2=1, X_3=0) = \frac{1}{3}$$

with all other combinations having probability 0.

Is there an X_4 such that X_1, X_2, X_3, and X_4 are exchangeable?

We'll assume that such an X_4 exists. Then, for example,

$$\mathbb{P}(X_1=0, X_2=1, X_3=1, X_4=0) = \mathbb{P}(X_1=0, X_2=0, X_3=1, X_4=1)$$

Now, $\mathbb{P}(X_1=0, X_2=1, X_3=1, X_4=0) = \mathbb{P}(X_1=0, X_2=1, X_3=1)$
$$- \mathbb{P}(X_1=0, X_2=1, X_3=1, X_4=1)$$
$$= 1/3 - \mathbb{P}(X_1=1, X_2=1, X_3=1, X_4=0) \text{ (by exchangeability)}$$

However,

$$\mathbb{P}(X_1=1, X_2=1, X_3=1, X_4=0) \leq \mathbb{P}(X_1=1, X_2=1, X_3=1) = 0$$

Hence, $\mathbb{P}(X_1=0, X_2=1, X_3=1, X_4=0) = 1/3$.

But,

$$\mathbb{P}(X_1=0, X_2=0, X_3=1, X_4=1) \leq \mathbb{P}(X_1=0, X_2=0, X_3=1) = 0$$

Thus, we have a contradiction and so X_1, X_2, X_3, X_4 are not exchangeable.

A finitely exchangeable sequence cannot necessarily be embedded into a larger finitely exchangeable sequence let alone an infinitely exchangeable one.

[Question Sheet Three Q2: consequence of finite exchangeability of $X_1, \dots, X_n$ is that $\mathrm{Corr}(X_i, X_j) \geq -\frac{1}{n-1}$ $i \neq j$. Consequently, infinitely exchangeable random variables are positively (non-negatively) correlated.].

THEOREM (DE FINETTI'S REPRESENTATION THEOREM FOR 0-1 RVs)

Let $X_1, X_2, \dots$ be a sequence of infinitely exchangeable 0-1 RVs (events), then the joint distribution of $X_1, \dots, X_n$ has an integral representation of the form

$$f(x_1, \dots, x_n) = \int_0^1 \left\{ \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i} \right\} f(\theta)\, d\theta$$

where $y_n = \sum_{i=1}^{n} x_i$ and $\theta = \lim_{n \to \infty} \frac{y_n}{n}$.

[Note: If $X_1, \dots, X_n$ are IID given θ then:

$$f(x_1, \dots, x_n) = \int_\theta f(x_1, \dots, x_n | \theta) f(\theta) \, d\theta$$

$$= \int_\theta \left\{ \prod_{i=1}^{n} f(x_i | \theta) \right\} f(\theta) \, d\theta \quad]$$

The interpretation of this theorem is of profound significance

① Conditional on a RV θ, the X_is are judged to be independent Bernoulli trials

MODEL/LIKELIHOOD $\quad f(x_i | \theta) = \theta^{x_i} (1-\theta)^{1-x_i}$

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta^{y_n} (1-\theta)^{n-y_n}$$

PRIOR ② θ is itself given a probability distribution $f(\theta)$

③ $\theta = \lim_{n \to \infty} \frac{y_n}{n}$ so that $f(\theta)$ represents beliefs about the limiting value of the mean of the X_is

Conditional upon θ, $X_1, \dots, X_n$ are a random sample from a Bernoulli with parameter θ generating a parameterised joint sampling distribution

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

when θ is assigned a prior distribution $f(\theta)$.

Note: θ is provided with a formal definition and the sentence "the true value of θ" has a well defined meaning.

THEOREM (General representation theorem - simplified form).

If $X_1, X_2, \ldots$ is an infinitely exchangeable sequence of RVs then the joint distribution of $X_1, \ldots, X_n$ has an integral representation of the form:

likelihood constructed of IID observations given θ — prior

$$f(x_1, \ldots, x_n) = \int_\theta \left\{ \prod_{i=1}^{n} f(x_i \mid \theta) \right\} f(\theta)\, d\theta$$

where θ is the limit of $n \to \infty$ of some function $g(x_1, \ldots, x_n)$ and $f(\theta)$ is a distribution over θ.