05/05/2022. LECTURE 23: NOTES MADE ON THE VISUALISER.

$$L(\theta, d) = \theta(\theta - d)^2, \quad \theta \sim \text{Gamma}(\alpha, \beta), \quad X_i \mid \theta \sim P_o(\theta)$$

a) Find the Bayes rule and Bayes risk for an immediate decision.

Consider $[\Theta, \mathcal{D} = \Theta, \pi(\theta), L(\theta, d) = \theta(\theta - d)^2]$

$$\rho(\pi, d) = E_{(\pi)}(L(\theta, d)) = E_{(\pi)}(\theta^3) - 2d E_{(\pi)}(\theta^2) + d^2 E_{(\pi)}(\theta)$$

$$\frac{d}{dd} E_{(\pi)}(L(\theta, d)) = -2E_{(\pi)}(\theta^2) + 2d E_{(\pi)}(\theta)$$

$$\Rightarrow d^* = \frac{E_{(\pi)}(\theta^2)}{E_{(\pi)}(\theta)}$$

Note as $\theta > 0$, $E_{(\pi)}(\theta) > 0$ and so d^* is a minimum.

$$\rho^*(\pi(\theta)) = E_{(\pi)}(\theta^3) - 2\frac{E_{(\pi)}(\theta^2)}{E_{(\pi)}(\theta)} E_{(\pi)}(\theta^2) + \left(\frac{E_{(\pi)}(\theta^2)}{E_{(\pi)}(\theta)}\right)^2 E_{(\pi)}(\theta)$$

$$= E_{(\pi)}(\theta^3) - \frac{E^2_{(\pi)}(\theta^2)}{E_{(\pi)}(\theta)}.$$

For the immediate decision, $\pi(\theta) = f(\theta)$, i.e. $\theta \sim \text{Gamma}(\alpha, \beta)$.

$$E(\theta^k) = \int_0^\infty \underbrace{\theta^k \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha - 1} \exp(-\beta\theta)}_{\text{kernel of Gamma}(\alpha+k, \beta) \text{ provided } \alpha + k > 0} d\theta$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \frac{\beta^\alpha}{\beta^{\alpha+k}} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \beta^{-k}.$$

Thus, $E(\theta) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \beta^{-1} = \frac{\alpha}{\beta}$

$$E(\theta^2) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \beta^{-2} = \frac{\alpha(\alpha+1)}{\beta^2}$$

$$E(\theta^3) = \frac{\Gamma(\alpha+3)}{\Gamma(\alpha)} \beta^{-3} = \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$$

So, solving $[\Theta, \mathcal{D} = \Theta, f(\theta), L(\theta, d) = \theta(\theta - d)^2]$ gives

$$d^* = \frac{E(\theta^2)}{E(\theta)} = \frac{\alpha(\alpha+1)}{\beta^2} \times \frac{\beta}{\alpha} = \frac{\alpha+1}{\beta}.$$

$$\rho^*(f(\theta)) = E(\theta^3) - \frac{E^2(\theta^2)}{E(\theta)}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3} - \left(\frac{\alpha+1}{\beta}\right)\frac{\alpha(\alpha+1)}{\beta^2}$$

$$= \frac{\alpha(\alpha+1)[(\alpha+2) - (\alpha+1)]}{\beta^3} = \frac{\alpha(\alpha+1)}{\beta^3}.$$

b. Find the Bayes rule and risk if we have a sample of size n.

Observe $x = (x_1, \dots, x_n)$ $\theta \sim \text{Gamma}(\alpha, \beta)$, $X_i | \theta \sim P_0(\theta)$

$[f(x_i|\theta) \propto \theta^{x_i} \exp(-\theta)]$

$$\theta | x \sim \text{Gamma}\left(\alpha + \sum_{i=1}^n x_i, \beta + n\right)$$

We can exploit conjugacy $\left[\alpha \mapsto \alpha + \sum_{i=1}^n x_i, \beta \mapsto \beta + n\right]$ so that the solution of $[\Theta, \mathcal{D} = \Theta, f(\theta|x), L(\theta, d) = \theta(\theta - d)^2]$ is

$$d^* = \frac{\alpha + \sum_{i=1}^{n} x_i + 1}{\beta + n} = \frac{\beta}{\beta+n}\left(\frac{\alpha+1}{\beta}\right) + \frac{n}{\beta+n}\,\bar{x}$$

Weighted average of the Bayes rule for the immediate decision and mle $\bar{x}$.

Note as $n \to \infty$, $d^* \to \bar{x}$

$$\rho^*(f(\theta|x)) = \frac{\left(\alpha + \sum_{i=1}^{n} x_i\right)\left(\alpha + \sum_{i=1}^{n} x_i + 1\right)}{(\beta+n)^3}$$

c) Find the Bayes risk of the sampling procedure for a sample of size n

Bayes decision function $\delta^*(X) = \dfrac{\alpha + \sum_{i=1}^{n} X_i + 1}{\beta+n}$ with

$$\rho_n^* = E[\rho^*(f(\theta|x))] = E\left[\frac{\left(\alpha + \sum_{i=1}^{n} X_i\right)\left(\alpha + \sum_{i=1}^{n} X_i + 1\right)}{(\beta+n)^3}\right]$$

Let $Y = \sum_{i=1}^{n} X_i$ then:

$$\rho_n^* = \frac{\alpha(\alpha+1) + (2\alpha+1)E(Y) + E(Y^2)}{(\beta+n)^3}$$

Now, $E(Y) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$

$$= \sum_{i=1}^{n} E(E(X_i|\theta)) \quad X_i|\theta \sim Po(\theta)$$

$$= \sum_{i=1}^{n} E(\theta) = \frac{n\alpha}{\beta} \quad \theta \sim Gamma(\alpha, \beta)$$

$$E(Y^2) = E(E(Y^2|\theta)) = E\left(Var(Y|\theta) + E^2(Y|\theta)\right)$$

Now, $Var(Y|\theta) = Var\left(\sum_{i=1}^{n} X_i \,|\, \theta\right) = \sum_{i=1}^{n} Var(X_i|\theta)$

(independence of the X_i's GIVEN θ)

$= n\theta$ as $X_i \mid \theta \sim Po(\theta)$

$$E(Y \mid \theta) = \sum_{i=1}^{n} E(X_i \mid \theta) = n\theta$$

Hence,

$$E(Y^2) = E\left(n\theta + n^2\theta^2\right) = n\frac{\alpha}{\beta} + n^2\frac{\alpha(\alpha+1)}{\beta^2}$$

Thus,

$$\rho_n^* = \frac{\alpha(\alpha+1) + (2\alpha+1)n\frac{\alpha}{\beta} + n\frac{\alpha}{\beta} + n^2\frac{\alpha(\alpha+1)}{\beta^2}}{(\beta+n)^3}$$

$$= \frac{\alpha(\alpha+1)}{\beta^2(\beta+n)^3}\left[\beta^2 + 2n\beta + n^2\right] = \frac{\alpha(\alpha+1)}{\beta^2(\beta+n)}.$$

NB. $n = 0$ Risk of the immediate decision

As n increases, ρ_n^* decreases

Suppose that each observation costs a fixed amount c, then we can find the optimal choice of n by considering the TOTAL RISK, the sum of the risk of the sampling procedure and the sample cost

i.e. $R_n = \rho_n^* + nc$

increases in n

decreases in n

and we can use this to find an optimal n.