Survival probabilities for branching Brownian motion with absorption

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Abstract
We study a branching Brownian motion (BBM) with absorption, in which particles move as Brownian motions with drift $-\rho$, undergo dyadic branching at rate $\beta > 0$, and are killed on hitting the origin. In the case $\rho > \sqrt{2\beta}$ the extinction time for this process, $\zeta$, is known to be finite almost surely. The main result of this article is a large-time asymptotic formula for the extinction probability $P^x(\zeta > t)$ in the case $\rho > \sqrt{2\beta}$, where $P^x$ is the law of the BBM with absorption started from a single particle at the position $x > 0$.

We also introduce an additive martingale, $V$, for the BBM with absorption, and then ascertain the convergence properties of $V$. Finally, we use $V$ in a ‘spine’ change of measure and interpret this in terms of ‘conditioning the BBM to survive forever’ when $\rho > \sqrt{2\beta}$, in the sense that it is the large $t$-limit of the conditional probabilities $P^x(A|\zeta > t + s)$, for $A \in \mathcal{F}_s$.

1 Introduction and summary of results

The object of study in this article is a (dyadic) branching Brownian motion with absorption. This is a branching process in which all particles diffuse spatially as Brownian motions with drift $-\rho$, where $\rho \in \mathbb{R}$, and are killed (removed from the process) on hitting the origin. All living particles undergo binary fission at exponential rate $\beta > 0$, with offspring moving off independently from their birth positions and repeating stochastically the behaviour of their parent, and so on. Let the configuration of particles at time $t$ be $\{Y_u(t) : u \in N_t^{-\rho}\}$, where $N_t^{-\rho}$ is the set of surviving particles. We shall refer to this process as a $(-\rho, \beta; \mathbb{R}^+)$-BBM, with probabilities $\{P^x : x \in \mathbb{R}\}$, where $P^x$ is the law of the process initiated from a single particle at $x > 0$.

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We define \( \zeta := \inf \{ t > 0 : N_t^{-\rho} = \emptyset \} \) to be the extinction time of the \((-\rho, \beta; \mathbb{R}^+)\text{-BBM}, so that \( \{ \zeta = \infty \} \) is understood to be the event that the process survives forever. The right-most particle in this process is defined to be \( R_t := \sup \{ Y_u(t) : u \in N_t^{-\rho} \} \) on \( \{ \zeta > t \} \), and 0 otherwise. Trivially, we note that \( \{ R_t > 0 \} = \{ \zeta > t \} \). We also introduce the notation \( N_t^{-\rho}(a,b) := \sum_{u \in N_t^{-\rho}} 1_{\{ Y_u(t) \in (a,b) \}}, \) where \( a, b \in [-\infty, \infty] \), for the number of surviving particles with spatial positions in the interval \((a,b)\) at time \( t \).

This model was studied in Harris et al. [12], where it was shown using martingale arguments that for all \( x > 0 \), \( P^x(\zeta < \infty) = 1 \) when \( \rho \geq \sqrt{2\beta} \), and further that \( P^x(\zeta < \infty) \in (0,1) \) when \( \rho < \sqrt{2\beta} \). The main result of this article gives an asymptotic expression for the (decaying) survival probability \( P^x(R_t > 0) \) when \( \rho > \sqrt{2\beta} \).

**Theorem 1.** For \( \rho > \sqrt{2\beta} \) and \( x > 0 \),

\[
\lim_{t \to \infty} P^x(R_t > 0) \frac{\sqrt{2\pi t x}}{\rho x} e^{-\rho x + (\frac{\beta x^2}{2} - \beta) t} = K, \tag{1}
\]

for some constant \( K > 0 \) that is independent of \( x \). This is equivalent to

\[
P^x(R_t > 0) \sim \frac{1}{2} \rho^{\frac{3}{2}} K \times E^x(N_t^{-\rho}(0,\infty)). \tag{2}
\]

This new result will be proved in Section 2 (although some of the details will be delayed until Section 3) and represents the bulk of the original work of this paper. Theorem 1 was motivated by the related results of Chauvin and Rouault [6, Theorem 1] (for standard BBM) and Harris et al. [12, Theorem 6] (for the \((-\rho, \beta; \mathbb{R}^+)\text{-BBM}), which gave asymptotic expressions for the probabilities \( P^x(R_t > \lambda t + \theta) \), where \( \theta \in \mathbb{R} \) and \( \lambda > 0 \) is chosen sufficiently large that \( P^x(R_t > \lambda t + \theta) \to 0 \) as \( t \to \infty \).

This paper also complements the work on survival probabilities for the \((-\rho, \beta; \mathbb{R}^+)\text{-BBM of Kesten [14], who considered the ‘critical’ case \( \rho = \sqrt{2\beta} \). (Kesten [14] actually allowed random numbers of offspring, but our dyadic branching results would generalise to this case without too much difficulty.)

Kesten showed that when \( \rho = \sqrt{2\beta} \), there exists a constant \( C \in (0,\infty) \), depending only on \( \beta \), such that for \( x > 0 \),

\[
x e^{\rho x - C_1(\log t)^2} \leq P^x(R_t > 0) \exp \left( \frac{3\rho^2 x^2}{2} \right)^{\frac{1}{2}} t^{\frac{3}{2}} \leq (1 + x) e^{\rho x - C_1(\log t)^2}.
\]

An important difference between Kesten’s result and Theorem 1 is that the survival probability no longer decays like the expected total number of particles. To see this, recall that for measurable \( f \), the ‘Many-to-One Lemma’ (see Hardy and Harris [9], for example) asserts here that

\[
E^x \sum_{u \in N_t^{-\rho}} f(Y_u(t)) = e^{3\beta x^2} E^x_{-\rho}(f(Y_t); \tau_0 > t), \tag{3}
\]
where $Y$ is a Brownian motion with drift $-\rho$ under the measure $\mathbb{P}^{x,\rho}$, with associated expectation operator $\mathbb{E}^{x,\rho}$, and $\tau_0 := \inf\{t > 0 : Y_t = 0\}$. From this it follows that $E^{x,\rho}N_t(0,\infty) = e^{\beta t}E^{x,\rho}(1_{\{\tau_0>0\}})$ is of order $t^{-\frac{1}{2}}$ when $\rho = \sqrt{2}\beta$, which is a slower decay in $t$ than $\mathbb{P}^{x}(R_t > 0)$. No precise asymptotic for $\mathbb{P}^{x}(R_t > 0)$ is known when $\rho = \sqrt{2}\beta$, although we hope to treat this case in future work.

We now define an additive martingale for the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

**Lemma 2.** For all $\beta, \rho > 0$ the process

$$V(t) := \sum_{u \in N_{-\rho}^t} Y_u(t)e^{\rho Y_u(t)+\left(\frac{1}{2}\rho^2 - \beta\right)t}$$

defines an additive martingale for the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

Proving this is an easy application of the Many-to-One Lemma and the branching Markov property. We can now use the martingale $V$ to change measure on the probability space of the $(-\rho, \beta; \mathbb{R}^+)$-BBM, and use a spine construction to describe the behaviour of the BBM under the new measure. A spine change of measure is one in which the law of the single initial particle (the spine) is altered, and all sub-trees branching off from the spine behave as if under the original law $\mathbb{P}$. We define a new measure $\mathbb{Q}^x$ on the same probability space as $\mathbb{P}^x$ via

$$\frac{d\mathbb{Q}^x}{d\mathbb{P}^x}igg|_{\mathcal{F}_s} = \frac{1}{x}e^{-\rho x} \sum_{u \in N_{-\rho}^s} Y_u(s)e^{\rho Y_u(s)+\left(\frac{1}{2}\rho^2 - \beta\right)s} = \frac{V(s)}{V(0)}.$$ 

For full details of the construction of such changes of measure, we refer the reader to the spine theory given in Hardy and Harris [9], but also see Chauvin and Rouault [6], Kyprianou [16], and references therein. We note that $V(t)$ may also be obtained on taking a suitable conditional expectation of a single-particle martingale for the spine only, in the manner of Hardy and Harris [9], and then in calculations following their methods, one can prove the following.

**Lemma 3.** Under $\mathbb{Q}^x$, the $(-\rho, \beta; \mathbb{R}^+)$-BBM can be reconstructed in law as:

- starting from position $x$, the initial ancestor diffuses as a Bessel-3 process;
- at rate $2\beta$ the initial ancestor undergoes fission producing two particles;
- one of these particles is selected uniformly at random;
- this chosen particle (the spine) repeats stochastically the behaviour of their parent;
- the other particle initiates from its birth position an independent copy of a $(-\rho, \beta; \mathbb{R}^+)$-BBM with law $\mathbb{P}$.

The next result describes the convergence properties of $V$, and is proved in Section 4.
Theorem 4. If $0 < \rho < \sqrt{2\beta}$, then $V$ is uniformly integrable and the events $\{V(\infty) > 0\}$ and $\{\zeta = \infty\}$ agree up to a $P^x$-null set. If $\rho \geq \sqrt{2\beta}$ then, $P^x$-almost surely, $V(\infty) = 0$.

Moreover it can be shown that changing measure with $V$ corresponds to conditioning the sub-critical process to ‘survive forever’, in the following sense.

Theorem 5. Let $\rho > \sqrt{2\beta}$. For $s > 0$ fixed and $A \in \mathcal{F}_s$, as $t \to \infty$

$$P^x(A|R_{s+t} > 0) \to Q^x(A).$$

Given the asymptotic expression for $P^x(R_t > 0)$ from Theorem 1, the proof of Theorem 5 is nearly unchanged from the proof of the analogous result Chauvin and Rouault [6, Theorem 4] for standard BBM, and thus omitted here.

Obtaining a spine construction by ‘conditioning’ on a null event, in the manner of Theorem 5, has been seen before for BBM in Chauvin and Rouault [6], and a related conditioning for the $(-\rho, \beta; \mathbb{R}^+)$-BBM appears in Harris et al. [12]; in both these cases, the conditioning event is $R_t > \lambda t + \theta$, for $\theta \in \mathbb{R}$ and suitably large $\lambda > 0$. Similar ideas for superprocesses arise in ‘immortal particle’ constructions — see, for example, Evans [8]. More generally, in recent years spine constructions have been seen to be a powerful technique in the analysis of branching processes, and have yielded intuitively simple, elegant, proofs of some important results. In particular, the reader is referred to Lyons et al. [18], Lyons [17], Kurtz et al. [15], Biggins and Kyprianou [2], Athreya [1], Kyprianou [16], Hardy and Harris [11], and references therein, for applications of the spine approach.

2 Survival probabilities

In our proof of Theorem 1 we use similar ideas to those seen in Chauvin and Rouault [6], involving links with nonlinear partial differential equations and the Brownian bridge. However, there are significant novelties here — in particular Proposition 6, and the study of the Brownian bridge conditioned to avoid the origin, which comes in Section 3.

Proof of Theorem 1. Define $u(t, x) := P^x(R_t > 0)$, and let $s \in [0, t]$. Then it follows from the branching Markov property that

$$u(t, x) = E^x\left(P^x(R_t > 0|\mathcal{F}_s)\right) = E^x\left(\prod_{u \in N_{s-r}} u(t-s, Y_u(s))\right),$$

whence $\prod_{u \in N_{s-r}} u(t-s, Y_u(s))$ is a product martingale on $[0, t]$, from which we have that $u \in C^{1,2}([\mathbb{R}^+ \times \mathbb{R}^+])$ and satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial u}{\partial x} + \beta u(1-u),$$
with initial condition \( u(0, x) = 1_{\{x > 0\}} \) for \( x > 0 \), and boundary condition \( u(t, 0) \equiv 0 \) for \( t > 0 \) (cf. Champneys et al. [5], or Chauvin and Rouault [6]). Hence for \( Y \) a Brownian motion (started at \( x > 0 \)) with drift \(-\rho\) under \( \mathbb{P}^x_{-\rho} \) and \( \tau_0 := \inf\{s > 0 : Y_s = 0\} \),

\[
M_t(s) := u(t - (s \wedge \tau_0), Y(s \wedge \tau_0)) \exp\left( \beta \int_0^{s \wedge \tau_0} (1 - u(t - \phi, Y(\phi))) \, d\phi \right)
\]
is a uniformly integrable \( \mathbb{P}^x_{-\rho} \)-martingale on \([0, t]\). As a consequence of the Optional-Stopping Theorem, we can write

\[
u(t, x) = \mathbb{E}^x_{-\rho}\left( 1_{\{Y_t > 0\}}e^{\beta \int_0^t (1 - u(t\wedge \tau_0 - \phi, Y_\tau(\phi))) \, d\phi}; \tau_0 > t \right).
\]

For later use, we note that applying Chernov’s inequality here gives an upper bound on \( u \):

\[
u(t, x) \leq e^{\beta t}\mathbb{E}^x_{-\rho}(1_{\{Y_t > 0\}}) \leq e^{\rho x - (\frac{1}{2} \rho^2 - \beta) t}, \quad \text{for all } t, x > 0. \tag{4}
\]

We now re-write our expression for \( u \) in terms of the Brownian bridge, giving

\[
u(t, x) = e^{\beta t} \int_0^\infty \mathbb{P}_{-\rho}(Y_t \in dz) \mathbb{E}\left( e^{-\beta \int_0^t u(t\wedge \tau_0 - \phi, B^t_{\rho}(\phi)) \, d\phi}; \tau_0^t > t \right),
\]

where \( \tau_0^t = \tau_0^t(x, z) := \inf\{\phi > 0 : B^t_{\rho}(\phi) = 0\} \) and \( \{B^t_{\rho}(\phi) : \phi \in [0, t]\} \) is the Brownian bridge from \( x \) to \( z \). Hence

\[
u(t, x) \frac{\sqrt{2\pi t^3}}{x e^{x^2}} e^{(\frac{1}{2} \rho^2 - \beta) t} = \int_0^\infty \frac{t e^{-\rho x} e^{-\frac{1}{2}(x-z)^2}}{x} e^{-\beta \int_0^t u(t\wedge \tau_0^t - \phi, B^t_{\rho}(\phi)) \, d\phi}; \tau_0^t > t \, dz. \tag{5}
\]

Using the distributional equivalence

\[
\int_0^t u(t - \phi, B^t_{\rho}(\phi)) \, d\phi \overset{d}{=} \int_0^t u(\phi, B^t_{\rho}(\phi)) \, d\phi,
\]

obtained by a time-reversal, and the explicit formula for the probability that the Brownian bridge avoids the origin, \( \mathbb{P}(\tau_0^t > t) = 1 - \exp\left( -\frac{2xz}{t} \right) \), we may re-write the right-hand side of (5) as

\[
\int_0^\infty \frac{t}{x} (1 - e^{-\frac{2xz}{t}}) e^{-\rho x} e^{-\frac{1}{2}(x-z)^2} \mathbb{E}\left( e^{-\beta \int_0^t u(\phi, B^t_{\rho}(\phi)) \, d\phi}; \tau_0^t > t \right) \, dz. \tag{6}
\]

Now as \( t \to \infty \), \( 0 \leq \frac{1}{t} (1 - e^{-\frac{2xz}{t}}) \uparrow 2z \), and so it is sufficient to show that for some function \( g : [0, \infty) \to (0, 1] \),

\[
\mathbb{E}\left( e^{-\beta \int_0^t u(\phi, B^t_{\rho}(\phi)) \, d\phi}; \tau_0^t > t \right) \xrightarrow{t \to \infty} g(z), \tag{7}
\]
since then by dominated convergence the expression given at equation (6) tends to
\(\int_0^\infty 2ze^{-\rho^2 g(z)} \, dz\) as \(t \to \infty\).

The intuition for the convergence of the conditional expectation to a function of \(z\) only becomes clear when we consider how the Brownian bridge behaves for large \(t\). We can represent the Brownian bridge \(B_{t,z,x}(\phi)\) as

\[ B_{t,z,x}(\phi) = Y(\phi) - \frac{\phi}{t} Y(t) + \frac{\phi}{t} (x - z) + z, \]

where \(Y\) is a standard Brownian motion started at the origin. Since
\(\frac{Y_t}{t} \to 0\) almost surely as \(t \to \infty\), for \(0 < \phi \ll t\) the bridge is approximately \(B_{t,z,x}(\phi) \approx z + Y(\phi)\), that is to say the conditioning event \(\{Y(t) \in dx\}\) does not significantly affect the Brownian motion for small \(\phi\). When we additionally condition the bridge to avoid the origin, we might expect that \(B_{t,z,x}(\phi) \approx X(\phi)\) for \(\phi \ll t\), where \(X\) is a Bessel process started at \(z\) (in fact, we will see later that a Brownian bridge conditioned to avoid the origin is equal in law to a Bessel-3 bridge with the same start and end points). To make this idea watertight we will show that
\(\{B_{t,z,x}(\phi) | \tau_0^t > t, \phi \geq 0\}\) converges in distribution to \(\{X(\phi) | \phi \geq 0\}\) as \(t \to \infty\). For all \(t > 0\) we define \(B_{t,z,x}(\phi) \equiv x\) for \(\phi > t\) to ensure that the conditioned processes have paths in \(D_{[0,\infty)}[0, \infty)\) — the set of càdlàg paths in \([0, \infty)\).

The exponential decay of \(u(t, x)\) with respect to \(t\) — recall equation (4) — allows us to essentially neglect the contribution from the tail of the integral to the conditional expectation in equation 7. This is vitally important, as it means that the limit as \(t \to \infty\) in the conditional expectation is independent of \(x\). In the next section we turn this intuitive explanation into a rigorous proof of the following proposition.

**Proposition 6.** For each \(z > 0\),

\[ \mathbb{E}\left(e^{-\beta \int_0^t u(\phi, B_{t,z,x}(\phi)) \, d\phi} \mid \tau_0^t > t\right) \xrightarrow{t \to \infty} \mathbb{E}_B\left(e^{-\beta \int_0^\infty u(\phi, X(\phi)) \, d\phi}\right) =: g(z) \in (0, 1], \]

where \(\{X(\phi) : \phi \geq 0\}\) is a Bessel-3 process under \(\mathbb{P}_B^z\).

This completes the proof of equation (1), and equation (2) now follows from the Many-to-One Lemma and the one-particle calculation

\[ \mathbb{P}_x(\tau_0 > t) \sim \frac{2}{\pi(\rho^2 t)^3} xe^{\rho x - (\frac{1}{2} \rho^2 - \beta) t} \quad \text{as} \quad t \to \infty. \]

\[ \square \]

### 3 Proof of Proposition 6

To simplify notation, we will use the family of measures indexed by \(t\), \(\mathbb{P}_t^{z,x}\), with associated expectation \(\mathbb{E}_t^{z,x}\), for the laws of the Brownian bridges of length \(t\) conditioned to avoid the origin (but remember that we have extended the
bridge definitions to include times $\phi > t$). For the remainder of this section we will just use the notation $\{X\}_{\phi \geq 0}$ for a process with paths in $D_{[0,\infty]}[0,\infty)$, and remember that under $\mathbb{P}_t^\phi$, $X$ is the conditioned Brownian bridge, while under $\mathbb{P}^\infty_B$, $X$ is a Bessel-3 process.

The main reference for the theory on weak convergence in this section is Ethier and Kurtz [7]. Denote by $D_S[0,\infty)$ the set of càdlàg paths in $S$. The Skorohod metric can be defined on $D_S[0,\infty)$, and this space is complete and separable with respect to the Skorohod metric.

**Lemma 7.** As $t \to \infty$, $\mathbb{P}_t^\phi \Rightarrow \mathbb{P}_B^\infty$.

The proof of Lemma 7 is given at the end of this section.

**Proof of Proposition 6.** Fix $T > 0$ and let $t > T$. For notational convenience we define, for $a, b \geq 0$ and $X$ a process with càdlàg paths in $D_{[0,\infty]}[0,\infty)$,

$$I(a, b) := \int_a^b u(\phi, X(\phi)) d\phi.$$

Now let $\varepsilon > 0$ and define

$$A(t) := \mathbb{P}_t^\infty \left( e^{-\beta I(0,T)} - \beta I(T,t); \sup_{T \leq \phi} X(\phi)/\phi \geq \varepsilon \right),$$

$$B(t) := \mathbb{P}_t^\infty \left( e^{-\beta I(0,T)} - \beta I(T,t); \sup_{T \leq \phi} X(\phi)/\phi \leq \varepsilon \right),$$

so that $\mathbb{P}_t^\infty(e^{-\beta I(0,t)}) = A(t) + B(t)$.

Now by Lemma 7 and Ethier and Kurtz [7, Theorem 3.1],

$$A(t) \leq \mathbb{P}_t^\infty \left( \sup_{T \leq \phi} X(\phi)/\phi > \varepsilon \right) \to \mathbb{P}_B^\infty \left( \sup_{T \leq \phi} X(\phi)/\phi > \varepsilon \right) \quad \text{as } t \to \infty.$$

Since $X(\phi)/\phi \to 0 \mathbb{P}_B^\infty$-almost surely as $t \to \infty$, the final probability in the line above can be made arbitrarily small (for any $\varepsilon > 0$) by letting $T \to \infty$.

To deal with the term $B(t)$, we bound it above and below with expressions that are equal (in the limit) to the required expectation as we first let $t \to \infty$, and then let $T \to \infty$. For the upper bound we have

$$B(t) \leq \mathbb{E}_t^\infty(e^{-\beta I(0,T)}) \to \mathbb{E}_B^\infty(e^{-\beta I(0,T)})$$

as $t \to \infty$. This follows from Lemma 7 because $e^{-\beta \int_0^T u(\phi, \cdot) d\phi}$ is a continuous bounded function of the sample paths. Letting $T \to \infty$, and using bounded convergence, we obtain

$$\limsup_{t \to \infty} B(t) \leq \mathbb{E}_B^\infty(e^{-\beta I(0,\infty)}). \quad (8)$$
For the lower bound, recall that $\rho > \sqrt{2/H}$, and so we can take $\varepsilon > 0$ sufficiently small that $\delta := -\rho \varepsilon + \left( \frac{1}{2} \rho^2 - \beta \right) > 0$. Then if $X(\phi) \leq \varepsilon \phi$, using the upper bound for $u(t, x)$ at equation (4) we have $u(\phi, X(\phi)) \leq e^{-\delta \phi}$ and hence

$$B(t) \geq \mathbb{E}_t^{z,x} \left( e^{-\beta I(0,T)} \sup_{T \leq \phi} \frac{X(\phi)}{\phi} \leq \varepsilon \right)$$

$$= \exp \left( -\frac{\beta}{\delta} (e^{-\delta T} - e^{-\delta t}) \right) \mathbb{E}_t^{\pi,z} \left( e^{-\beta I(0,T)} \sup_{T \leq \phi} \frac{X(\phi)}{\phi} \leq \varepsilon \right)$$

$$\to \exp \left( -\frac{\beta}{\delta} e^{-\delta T} \right) \mathbb{E}_B^{\pi,0} \left( e^{-\beta I(0,T)} \sup_{T \leq \phi} \frac{X(\phi)}{\phi} \leq \varepsilon \right)$$

as $t \to \infty$. Note that, although $e^{-\beta \int_0^\infty u(\phi, X(\phi)) \, d\phi} 1_{\sup_{T \leq \phi} \frac{X(\phi)}{\phi} \leq \varepsilon}$ is not continuous as a function of $v(\cdot) \in D_{[0,\infty][0,\infty]}$, the set of discontinuities,

$$\left\{ v(\cdot) \in D_{[0,\infty][0,\infty]} : \sup_{T \leq \phi} \frac{v(\phi)}{\phi} = \varepsilon \right\},$$

has $\mathbb{P}_B$-measure zero, and so the expectation does converge (see Billingsley [3, Theorem 5.1]). On letting $T \to \infty$, using bounded convergence we have

$$\liminf_{t \to \infty} B(t) \geq \mathbb{E}_B^{\pi,0}(e^{-\beta I(0,\infty)}),$$

and it follows from (8) and (9) that $\lim_{t \to \infty} B(t) = \mathbb{E}_B^{\pi,0}(e^{-\beta I(0,\infty)})$. Recalling that $A(t) \to 0$ as $t \to \infty$, we have shown that

$$\mathbb{E}_t^{z,x} \left( e^{-\beta \int_0^\infty u(\phi, X(\phi)) \, d\phi} \right) \xrightarrow{t \to \infty} \mathbb{E}_B^{\pi,0} \left( e^{-\beta \int_0^\infty u(\phi, X(\phi)) \, d\phi} \right),$$

as required.

It remains to show that the limit in the line above is strictly positive. Since the limit is bounded in $[0, 1]$, it is sufficient to prove a $\mathbb{P}_B^{\pi,\pi}$-almost sure domination of $\int_0^\infty u(\phi, X(\phi)) \, d\phi$ by some finite quantity. By the law of large numbers, there exists a random $T_0 < \infty$ such that $\mathbb{P}_B^{\pi,\pi}$-almost surely, for all $\phi > T_0$, $X(\phi)/\phi < \varepsilon$. Then $\mathbb{P}_B^{\pi}$-almost surely $u(\phi, X(\phi)) \leq e^{-\delta \phi}$ for all $\phi > T_0$, and as $u(\phi, X(\phi)) \leq 1$ for $0 \leq \phi \leq T_0$ we have an almost sure domination.

The remainder of this section is devoted to the proof of Lemma 7. We will first give a criterion for tightness of the measures $\mathbb{P}_t^{\pi,\pi}$. We state the condition for tightness of measures on $D_S[0,\infty]$ in terms of the following modulus of continuity. For $v(s) \in D_S[0,\infty]$, $\delta > 0$, and $T > 0$, define

$$w'(v, \delta, T) := \inf_{\{t_i\}} \max_{\{r_i\}} \sup_{r \in [t_i, t_{i+1}]} d(v(r), v(s)),$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \ldots < t_m = T$ with $m \geq 1$ and $\min_{1 \leq i \leq m} (t_i - t_{i-1}) > \delta$.

For a compact set $K \subset S$, let $K^\varepsilon$ denote the $\varepsilon$-expansion of $K$, that is to say the set $\{ x \in S : \inf_{y \in K} d(x, y) < \varepsilon \}$.
Theorem 8 (Ethier and Kurtz [7, Theorem 7.2]). Let \((S, d)\) be complete and separable, and let \(\{P_\alpha\}\) be a family of laws of processes with sample paths in \(D_S[0, \infty)\). Then \(\{P_\alpha\}\) is relatively compact if and only if the following two conditions hold.

(i) For every \(\varepsilon > 0\) and rational \(T \geq 0\), there exists a compact set \(\Gamma_{\varepsilon,T} \subset S\) such that
\[
\inf_\alpha P_\alpha(X(T) \in \Gamma_{\varepsilon,T}) \geq 1 - \varepsilon.
\]

(ii) For every \(\varepsilon > 0\) and \(T > 0\), there exists a \(\delta > 0\) such that
\[
\sup_\alpha P_\alpha(w'(X, \delta, T) \geq \varepsilon) \leq \varepsilon.
\]

(Recall that tightness and relative compactness are equivalent in complete separable spaces.)

Observe that both the conditions in Theorem 8 involve only the paths of the processes on fixed intervals \([0, T]\). Our strategy for proving tightness of the conditioned Brownian bridges rests on the fact that the conditions of Theorem 8 are satisfied by the Bessel process, and that for large enough \(t\), the process \(X\) under the law \(P_0^x\) is sufficiently ‘close’ in law to the Bessel process on \([0, T]\) for the conditions to hold for it also.

There two steps to the proof of Lemma 7: direct calculation with the transition densities shows that convergence holds in the sense of finite dimensional distributions, and then tightness of the measures implies full convergence in distribution.

Proof of Lemma 7: convergence of the finite dimensional distributions. We define the standard Brownian transition density (with respect to Lebesgue measure) to \(p_s(y_1, y_2)\), for \(s > 0\) and \(y_1, y_2 \in \mathbb{R}\), and then
\[
q_s(y_1, y_2) := p_s(y_1, y_2) - p_s(y_1, -y_2)
\]
is the transition density for Brownian motion killed at the origin. With this notation, the transition density for a Bessel-3 process is \(\frac{2s}{y_1} q_s(y_1, y_2)\). Now for any finite set of times \(\{t_1, \ldots, t_k\}\), we can re-write the \(P_0^y\)-law of \((X(t_1), \ldots, X(t_k))\) in terms of a standard Brownian motion \(Y\) (started at \(z\)) under the law \(P_0^z\), conditioned on its position at time \(t\) — we remark that rigorous justification for this slight abuse of notation can be found in Revuz and Yor [19, Chapter XI]. See also Borodin and Salminen [4, IV.20–IV.26] for some very similar calculations.

\[
P_0^y(X(t_1) \in dy_1; \ldots; X(t_k) \in dy_k) = \frac{P_0^y(Y(t_1) \in dy_1; \ldots; Y(t_k) \in dy_k; \tau_0 > t; Y(t) = x)}{P_0^y(\tau_0 > t; Y(t) = x)} = \frac{P_0^y(Y(t_1) \in dy_1; \ldots; Y(t_k) \in dy_k; Y(t) = x; \tau_0 > t)}{P_0^y(Y(t) = x; \tau_0 > t)},
\]
and then, by the Markov property, this probability density is equal to

\[
\frac{q_t(z, y_1) \cdots q_{t_k}(y_{k-1}, y_k) q_{t_k}(y_k, x)}{q_t(z, x)}.
\]

From this we conclude that the finite dimensional distributions of the Brownian bridge conditioned to avoid the origin are the finite dimensional distributions of the Bessel-3 bridge, and hence these two bridges are equal in law. This means that we can use \( \mathbb{P}_{t}^{z,x} \) for the law of a Bessel-3 bridge from \( z \) to \( x \) over the time interval \([0, t]\), with the extension \( X(\phi) \equiv x \) for \( \phi > t \). Further, a calculation with the explicit expressions for the transition densities shows that

\[
\frac{z \cdot q_{t-k}(y_k, x)}{y_k} q_{t}(z, x) = \frac{z}{y_k} \frac{\exp \left( -\frac{(y_k - x)^2}{2(t-k)} \right) \left( 1 - e^{-\frac{y_k}{2} - \frac{x}{2}} \right)}{\exp \left( -\frac{(z-x)^2}{2t} \right) \left( 1 - e^{-\frac{z}{2}} \right)} \to 1
\]
as \( t \to \infty \), and so

\[
\mathbb{P}_{t}^{z,x}(X(t_1) \in dy_1; \ldots; X(t_k) \in dy_k) \to \mathbb{P}_{B}^{z}(X(t_1) \in dy_1; \ldots; X(t_k) \in dy_k)
\]
as \( t \to \infty \). Hence the Bessel bridge from \( z \) to \( x \) on time interval \([0, t]\) converges in the sense of finite dimensional distributions to a Bessel process started at \( z \) as \( t \to \infty \).

**Proof of Lemma 7: tightness.** We break this proof down into a series of lemmas. The first lemma expresses formally the the intuitive notion that, for \( 0 < T < t \), the Bessel bridge behaves almost like a Bessel process on \([0, T]\).

**Lemma 9 (Revuz and Yor [19, Chapter XI, Exercise(3.10)])**. Fix \( T > 0 \), let \( t > T \), and define the \( \sigma \)-algebra \( \mathcal{F}_T := \sigma(X_s : 0 \leq s \leq T) \). \( \mathbb{P}_{t}^{z,x} \) has a density \( M^{(T)}(t) \) on \( \mathcal{F}_T \) with respect to \( \mathbb{P}_{B}^{z} \), given by

\[
\left. \frac{\text{d}\mathbb{P}_{t}^{z,x}}{\text{d}\mathbb{P}_{B}^{z}} \right|_{\mathcal{F}_T} = M^{(T)}(t) = \frac{\mathbb{P}_{B}^{z}(X(t) = x|\mathcal{F}_T)}{\mathbb{P}_{B}^{z}(X(t) = x)}.
\]

As \( t \to \infty \), \( M^{(T)}(t) \to 1 \) point-wise and in \( L^1(\mathbb{P}_{B}^{z}) \).

**Proof.** Let \( A \in \mathcal{F}_T \).

\[
\mathbb{P}_{t}^{z,x}(A) = \mathbb{P}_{B}^{z}(A|X(t) = x) = \frac{\mathbb{P}_{B}^{z}(1_A \mathbb{P}_{B}^{z}(X(t) = x|\mathcal{F}_T))}{\mathbb{P}_{B}^{z}(X(t) = x)};
\]

and then

\[
M^{(T)}(t) = \frac{\mathbb{P}_{B}^{z}(X(t) = x|\mathcal{F}_T)}{\mathbb{P}_{B}^{z}(X(t) = x)} = \frac{z \cdot q_{t-T}(X(T), x)}{X(T) \cdot q_{t}(z, x)}.
\]

Another calculation with the transition densities shows that this converges to 1 point-wise as \( t \to \infty \). Since \( \mathbb{P}_{B}^{z}(M^{(T)}(t)) = 1 \) for all \( t > T \), \( M^{(T)}(t) \to 1 \) in \( L^1(\mathbb{P}_{B}^{z}) \) also. \( \square \)
Lemma 10. Retaining the notation of the previous result, let \( A \in \mathcal{F}_T \). As \( t \to \infty \), \( P^z_x(A) \to P^z_B(A) \).

Proof. Note first that, by Lemma 9, \( 1_A M(T)(t) \to 1_A \) almost surely with respect to \( P^z_B \) as \( t \to \infty \). Also \( 1_A M(T)(t) \leq M(T)(t) \), and we now bound \( M(T)(t) \) uniformly in \( X(T) \). Noting that \( 1 - e^{-z} \leq x \), we have

\[
M(T)(t) = \frac{z}{X(T)} \frac{q_{z-T}X(T,x)}{q_{z}(z,x)}
\]

\[
= \frac{z}{X(T)} \sqrt{\frac{t}{t-T}} \exp\left(-\frac{(z-T)^2}{2(t-T)}\right) \left(1 - e^{-\frac{2z}{1-T}}\right)
\]

\[
\leq \frac{2xz}{t-T} \sqrt{\frac{t}{t-T}} \left(1 - e^{-\frac{2z}{1-T}}\right) \left(1 - 2e^{-\frac{2z}{1-T}}\right)^{-1}
\]

\[
\to 1 \quad \text{as} \quad t \to \infty.
\]

This deterministic bound is continuous on the interval \([T + \epsilon, \infty)\), for any \( \epsilon > 0 \), and so there exists a constant \( 0 < C(x, z) < \infty \) such that \( M(T)(t) \leq C(x, z) \) on \([T + \epsilon, \infty)\). Bounded convergence now finishes the argument.

Alternatively, since \( 1_A M(T)(t) \leq M(T)(t) \) and \( P^z_B(M(T)(t)) = 1 \) for all \( t > T \), dominated convergence (as stated in Kallenberg [13, Theorem 1.21]) gives

\[
\lim_{t \to \infty} P^z_t(A) = \lim_{t \to \infty} P^z_B(1_A M(T)(t)) = P^z_B(A).
\]

Since the Bessel-3 process has continuous paths, it can be easily checked that the conditions of Theorem 8 hold for the single law \( P^z_B \); another way to see this is that (Ethier and Kurtz [7, Lemma 2.1]) a single measure on a complete separable space is tight, and so \( P^z_B \) satisfies the conditions of Theorem 8. That is to say: for every \( \epsilon > 0 \) and rational \( T \geq 0 \), we can choose a compact set \( \Gamma_{\epsilon, T} \subset [0, \infty) \) such that

\[
P^z_B(X(T) \in \Gamma_{\epsilon, T}) \geq 1 - \frac{\epsilon}{2};
\]

(10)

for every \( \epsilon > 0 \) and \( T > 0 \), we can choose a \( \delta > 0 \) such that

\[
P^z_B(w'(X, \delta, T) \geq \epsilon) \leq \frac{\epsilon}{2}.
\]

(11)

The events \( \{X(T) \in \Gamma_{\epsilon, T}\} \) and \( \{w'(X, \delta, T) \geq \epsilon\} \) are both \( \mathcal{F}_T \)-measurable, and it follows from Lemma 10, and equations (10) and (11), that there exists a \( t_0(z) < \infty \) such that, for all \( t > t_0(z) \),

\[
P^z_t(X(T) \in \Gamma_{\epsilon, T}) \geq 1 - \epsilon \quad \text{and} \quad P^z_t(w'(X, \delta, T) \geq \epsilon) \leq \epsilon.
\]

Here \( \Gamma_{\epsilon, T} \) and \( \delta \) are those chosen at, respectively, (10) and (11). Theorem 8 now gives us that the family of laws \( \{P^z_{t\cdot}\}_{t>0(z)} \) is tight, and this is sufficient to complete the proof of Lemma 7. \( \square \)
4 Convergence properties of $V$

The uniform-integrability assertion of Theorem 4 will follow from the following, slightly stronger, convergence result.

**Proposition 11.** For $x > 0$ and any $p \in (1, 2]$, 

(i) the martingale $V$ is $L^p(P^x)$-convergent if $p \rho^2 / 2 < \beta$;

(ii) almost surely under $P^x$, $V(\infty) = 0$ when $\rho \geq \sqrt{2\beta}$.

We do not give a proof of this result here: it is a simple adaptation of the spine approach to $L^p$ convergence described in Hardy and Harris [10].

**Proof of Theorem 4.** If $\rho \geq \sqrt{2\beta}$ then, by Proposition 11, $V(\infty) = 0$ almost surely.

If $\rho < \sqrt{2\beta}$ then it follows from Proposition 11 that there exists a $p > 1$ such that $V$ converges in $L^p(P^x)$, and so, by Doob’s $L^p$ inequality, $V$ is uniformly integrable. It remains to show that $V(\infty) > 0$ almost surely.

To prove that $P^x(V(\infty) = 0; \zeta = \infty) = 0$ when $\rho < \sqrt{2\beta}$, we note that

$$P^x(V(\infty) = 0) = P^x(V(\infty) = 0; \zeta = \infty) + P^x(V(\infty) = 0; \zeta < \infty)$$

and hence it suffices to show that $P^x(V(\infty) = 0) = P^x(\zeta < \infty)$.

To show this, we make use of the following result on existence and uniqueness for solutions of the ‘one-sided FKPP equation’, proved in Harris et al. [12].

**Theorem 12.** The system

$$\frac{1}{2}f'' - \rho f' + \beta(f^2 - f) = 0 \text{ on } (0, \infty)$$

$$f(0^+) = 1$$

$$f(\infty) = 0$$

has a unique solution in $\{f \in C^2(0, \infty) : 0 \leq f(x) \leq 1, \forall x \in (0, \infty)\}$ if and only if $-\infty < \rho < \sqrt{2\beta}$, in which case the unique solution is $f(x) = P^x(\zeta < \infty)$.

If $\rho \geq \sqrt{2\beta}$ there is no solution to the system (12).

We now show that $p(x) := P^x(V(\infty) = 0)$ is a solution to (12) when $0 < \rho < \sqrt{2\beta}$. We have

$$p(x) = E^x \left( P^x(V(\infty) = 0|F_t) \right) = E^x \left( \prod_{a \in N^-_t} p(Y_a(t)) \right),$$

whence $p(x)$ satisfies the travelling-wave ODE. Since extinction in a finite time guarantees that $V(\infty) = 0$, we also have $\lim_{x \downarrow 0} p(x) = 1$. Considering the process path-wise, we see that increasing $x$ increases the value of $V$ under the...
law $P^x$. Recall here that $\rho > 0$, so $xe^{\rho x}$ is increasing in $x$. Hence $p(x)$ is monotone decreasing in $x$ and $p(x) \downarrow p(\infty)$ as $x \to \infty$.

Now consider taking any fixed infinite BBM tree started at $x$. For any fixed time $t > 0$, we have $N_t^{-\rho} \uparrow N^{-\rho}$ as $x \to \infty$. Looking at the process path-wise again, for all $u \in N_t^{-\rho}$ we have $Y_u(t) \uparrow \infty$ as $x \to \infty$. Taking the limit $x \to \infty$ in (13) we then have $p(\infty) = \mathbb{E}^0 \left( \prod_{u \in N_t^{-\rho}} p(\infty) \right)$, whence $p(\infty) \in \{0, 1\}$ and now uniform integrability of $V$ forces $p(\infty) = 0$. Uniqueness of the one-sided travelling wave (Theorem 12) now finishes the argument. □

References


