Proof of Stokes' Theorem (not examinable)

Lemma. *Let $\boldsymbol{r} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a continuously differentiable parametrisation of a smooth surface $\mathcal{S} \subset \mathbb{R}^3$. Suppose that the vector field $\boldsymbol{F}$ is continuously differentiable (in a neighbourhood of $\mathcal{S}$). Then*

$$\mathrm{curl}\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) \;=\; \frac{\partial \boldsymbol{F}}{\partial u}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial v} \;-\; \frac{\partial \boldsymbol{F}}{\partial v}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial u}\,. \tag{1}$$

Proof. Expanding the left hand side of (1) we find that

$$\mathrm{curl}\boldsymbol{F} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)$$

$$+ \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)$$

Collecting all terms in this expansion that contain F_1 we get

$$\left(\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial x}{\partial u} = \frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u}$$

where the last identity follows from the chain rule by adding and subtracting $\dfrac{\partial F_1}{\partial x}\dfrac{\partial x}{\partial u}\dfrac{\partial x}{\partial v}$.

Similarly, collecting all terms that contain F_2 and F_3 in the above expansion and proceeding in the same way, it follows that

$$\mathrm{curl}\boldsymbol{F} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) = \left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial F_2}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial F_3}{\partial u}\frac{\partial z}{\partial v}\right) - \left(\frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial F_2}{\partial v}\frac{\partial y}{\partial u} + \frac{\partial F_3}{\partial v}\frac{\partial z}{\partial u}\right)$$

which is equal to the right hand side of (1). □

With the help of this lemma and Theorem 3.7 we can now prove Stokes' Theorem.

Proof of Theorem 3.11.
(Only for the case where the parametrisation $\boldsymbol{r} : D \to \mathbb{R}^3$ of $\mathcal{S}$ is <u>twice</u> continuously differentiable and where the domain $D \subset \mathbb{R}^2$ satisfies the assumptions of Theorem 3.7.)

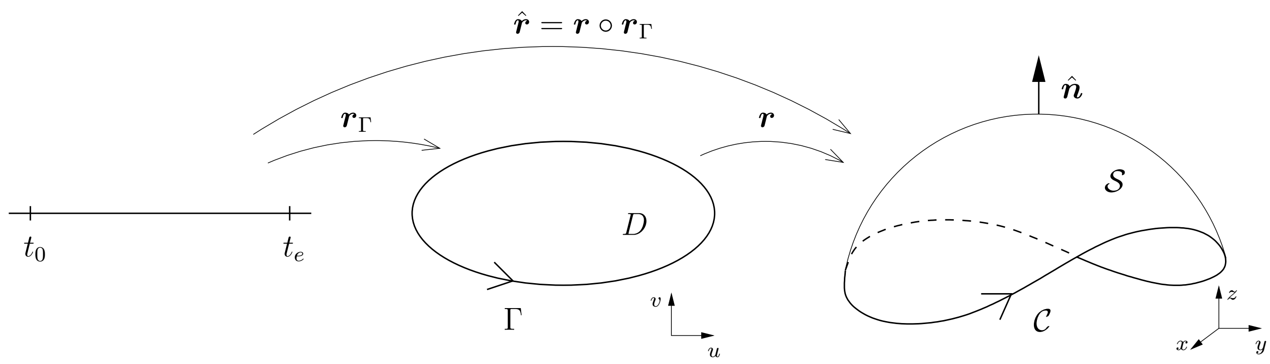
Let

$$\mathcal{S} := \{\boldsymbol{r}(u,v) : (u,v)^T \in D\} \qquad \text{and} \qquad \hat{\boldsymbol{n}} := \left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) \Big/ \left|\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right|\,.$$

Furthermore, let $\boldsymbol{r}_\Gamma : [t_0, t_e] \to \mathbb{R}^2$ be a parametrisation of the boundary Γ of D (described in the anticlockwise sense) with components $u(t)$ and $v(t)$, i.e. $\boldsymbol{r}_\Gamma(t) = (u(t), v(t))^T$.

The composition $\hat{\boldsymbol{r}} := \boldsymbol{r} \circ \boldsymbol{r}_\Gamma$ of the parametrisation of $\mathcal{S}$ with the parametrisation of Γ gives a parametrisation of $\mathcal{C}$, i.e.

$$\mathcal{C} = \{\hat{\boldsymbol{r}}(t) : t \in [t_0, t_e]\}, \qquad \text{where} \qquad \hat{\boldsymbol{r}}(t) := \boldsymbol{r}(\boldsymbol{r}_\Gamma(t)) = \boldsymbol{r}(u(t), v(t))\,.$$



Furthermore, $\mathcal{S}$ and $\mathcal{C}$ are correspondingly orientated. (No proof. Left as **exercise**!)

Also, using the chain rule

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{dv}{dt} . \tag{2}$$

By (1)

$$\begin{aligned}
\iint_{\mathcal{S}} \mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{dS} &= \iint_{\mathcal{S}} \mathrm{curl}\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) du\, dv = \\
&= \iint_{\mathcal{S}} \left[\frac{\partial \boldsymbol{F}}{\partial u}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial v} - \frac{\partial \boldsymbol{F}}{\partial v}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial u}\right] du\, dv = \\
&= \iint_{\mathcal{S}} \left[\frac{\partial}{\partial u}\left(\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial u}\right)\right] du\, dv
\end{aligned}$$

and therefore using Green's Theorem in the plane with

$$\Phi_1(u,v) := \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial u} \qquad \text{and} \qquad \Phi_2(u,v) := \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \frac{\partial \boldsymbol{r}}{\partial v}$$

we get

$$\begin{aligned}
\iint_{\mathcal{S}} \mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{dS} &= \iint_{\mathcal{S}} \left(\frac{\partial \Phi_2}{\partial u} - \frac{\partial \Phi_1}{\partial v}\right) du\, dv = \oint_{\Gamma} (\Phi_1 \boldsymbol{i} + \Phi_2 \boldsymbol{j}) \cdot d\boldsymbol{r} = \\
&= \int_{t_0}^{t_e} \left(\Phi_1(u(t),v(t))\,\frac{du}{dt} + \Phi_2(u(t),v(t))\,\frac{dv}{dt}\right) dt = \\
&= \int_{t_0}^{t_e} \boldsymbol{F}(\boldsymbol{r}(u(t),v(t))) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{dv}{dt}\right) dt
\end{aligned}$$

And finally using (2) and the fact that $\hat{\boldsymbol{r}}(t) = \boldsymbol{r}(u(t),v(t))$

$$\iint_{\mathcal{S}} \mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{dS} = \int_{t_0}^{t_e} \boldsymbol{F}(\hat{\boldsymbol{r}}(t)) \cdot \frac{d\hat{\boldsymbol{r}}}{dt}\, dt = \oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} .$$

□