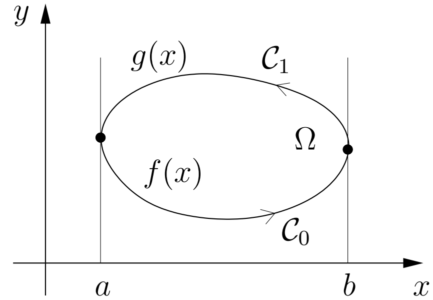
Proof of Green's Theorem in the Plane (not examinable)

Proof of Theorem 3.7. (Only for $\mathcal{C}$ smooth and semi-convex.)

As in the proof for the Divergence Theorem, since $\mathcal{C}$ is semi-convex, we can find two functions $f(x)$ and $g(x)$ with $f(x) \le g(x)$, as shown in the figure on the right. Hence, $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$, where



$$\begin{aligned} \mathcal{C}_0 &:= \{(x, f(x))^T : x \in [a,b]\} \\ -\mathcal{C}_1 &:= \{(x, g(x))^T : x \in [a,b]\} \qquad \text{(orientation!)} \end{aligned}$$

Thus

$$d\boldsymbol{r} \;=\; \frac{d\boldsymbol{r}}{dx}\,dx \;=\; \begin{cases} \Big(\boldsymbol{i} + f'(x)\boldsymbol{j}\Big)\,dx & \text{on } \mathcal{C}_0\ , \\ \Big(\boldsymbol{i} + g'(x)\boldsymbol{j}\Big)\,dx & \text{on } -\mathcal{C}_1\ . \end{cases}$$

(i) Let us first show that

$$\oint_{\mathcal{C}} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r} \;=\; \int_a^b \Big(\Phi_1(x, f(x)) - \Phi_1(x, g(x))\Big)\,dx\ . \tag{1}$$

Using Remark 1.24(b) and Remark 1.11(a) we have

$$\begin{aligned} \oint_{\mathcal{C}} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r} &= \int_{\mathcal{C}_0} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r} + \int_{\mathcal{C}_1} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r} = \int_{\mathcal{C}_0} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r} - \int_{-\mathcal{C}_1} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r} = \\ &= \int_a^b \Phi_1(x, f(x))\,dx - \int_a^b \Phi_1(x, g(x))\,dx \end{aligned}$$

which is equal to the r.h.s. of (1).

(ii) Now we show that

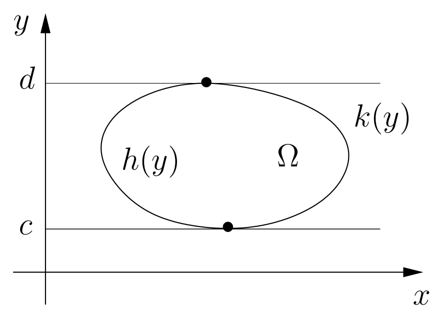
$$-\iint_{\Omega} \frac{\partial \Phi_1}{\partial y}\,dx\,dy \;=\; \oint_{\mathcal{C}} \Phi_1 \boldsymbol{i} \cdot d\boldsymbol{r}\ . \tag{2}$$

By the Fundamental Theorem of Calculus

$$\iint_{\Omega} \frac{\partial \Phi_1}{\partial y}\,dx\,dy \;=\; \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial \Phi_1}{\partial y}\,dy\right] dx \;=\; \int_a^b \Big(\Phi_1(x, g(x)) - \Phi_1(x, f(x))\Big)\,dx$$

which together with (1) establishes (2).

(iii) Similarly, by using y to parametrise we can find functions $h(y)$ and $k(y)$ with $h(y) \le k(y)$, as shown in the figure on the right, and establish in the same way as above that



$$\iint_{\Omega} \frac{\partial \Phi_2}{\partial x}\,dx\,dy \;=\; \oint_{\mathcal{C}} \Phi_2 \boldsymbol{j} \cdot d\boldsymbol{r}\ . \tag{3}$$

Finally, adding (2) and (3) we obtain (3.8). $\square$

Note. This proof can again be extended in a straightforward way to regions $\Omega = \bigcup_{i=1}^n \Omega_i$ where the boundary $\partial\Omega_i$ is smooth and semi-convex for all $i = 1, \ldots, n$.