Chapter 5

Partial Differential Equations

Definition 5.1. Any differential equation containing partial derivatives with respect to **at least two** different variables is called a **partial differential equation (PDE)**.

Note. The unknown function in any PDE must be a function of **at least two** variables, otherwise partial derivatives would not arise.

5.1 Classification of Partial Differential Equations

Definition 5.2 (as for ODEs). (a) The **order of a PDE** is equal to the order of the highest partial differential coefficient occurring in it.

(b) A PDE is **linear**, if the unknown function and its partial derivatives occur only to the first degree and if no products of the function and its derivatives occur.

(c) A PDE is **homogeneous**, if each term contains either the function or one of its partial derivatives.

Example 5.3. (a)

(b)

We will only study **linear homogeneous 2$^{\text{nd}}$-order PDEs of functions in two variables**, i.e.

$$a\,\frac{\partial^2 u}{\partial x^2} \;+\; 2h\,\frac{\partial^2 u}{\partial x \partial y} \;+\; b\,\frac{\partial^2 u}{\partial y^2} \;+\; c\,\frac{\partial u}{\partial x} \;+\; d\,\frac{\partial u}{\partial y} \;+\; e\,u \;=\; 0 \tag{5.1}$$

where a, b, c, d, e, and h are either constants or functions of x and y. They

- are very important in the description of many physical phenomena,
- contain many of the important features of typical PDEs in applications.

For other PDEs see [**MA30044**].

Theorem 5.4 (superposition principle). *If* $u_1, u_2, \ldots, u_n$ *are solutions of (5.1) and* $c_1, c_2, \ldots, c_n$ *are constants, then* $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ *is also a solution of (5.1).*

Proof. Follows directly from the linearity of (5.1). [**Exercise**]. □

The form of (5.1) resembles that of a general conic section, i.e.

$$a\,x^2 \;+\; 2h\,xy \;+\; b\,y^2 \;+\; c\,x \;+\; d\,y \;+\; e \;=\; 0.$$

In analogy to conic sections we have

Definition 5.5. The PDE is of $\left.\begin{array}{c}\text{elliptic}\\ \text{parabolic}\\ \text{hyperbolic}\end{array}\right\}$ type when $\left\{\begin{array}{l} ab - h^2 > 0,\\ ab - h^2 = 0,\\ ab - h^2 < 0.\end{array}\right.$

In the following, we will look at one important representative for each of these types and solve them by separation of variables, varying the domain $D \in \mathbb{R}^2$ and the boundary conditions.

5.2 Separation of Variables – Fourier's Method

5.2.1 Elliptic PDEs – Laplace's Equation

$$\nabla^2 u \;=\; \frac{\partial^2 u}{\partial x^2} \;+\; \frac{\partial^2 u}{\partial y^2} \;=\; 0\,. \tag{5.2}$$

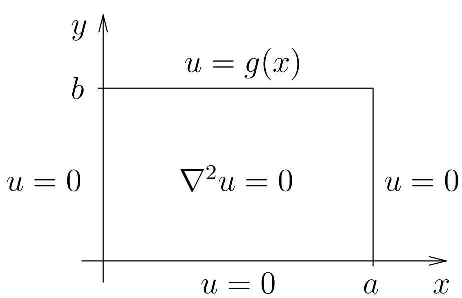
Laplace's equation (5.2) takes the general form (5.1) with $a = b = 1$ and $h = 0$. Therefore $ab - h^2 = 1 > 0$ and (5.2) is **elliptic**.

Recall Example 2.16.

Other applications: fluid mechanics, elasticity, etc.

Let $D := \{(x, y) \in \mathbb{R}^2 : 0 < x < a \text{ and } 0 < y < b\}$, i.e. a rectangular domain in $\mathbb{R}^2$.

BVP1: Find a solution $u : D \to \mathbb{R}$ of (5.2) on D which satisfies the **(Dirichlet) boundary conditions**:



$$\left.\begin{array}{l}\left.\begin{array}{rcl} u(0,y) &=& 0\\ u(a,y) &=& 0\end{array}\right\} \text{ for } 0 < y < b\\[1em] \left.\begin{array}{rcl} u(x,0) &=& 0\\ u(x,b) &=& g(x)\end{array}\right\} \text{ for } 0 < x < a\end{array}\right\} \tag{5.3}$$

where $g : (0, a) \to \mathbb{R}$ is a given function of x.

Step 1: Separation of variables.

Assume that $u(x,y) = X(x)Y(y)$ is a solution of (5.2). Then (5.2) becomes

Therefore

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda \quad (= const), \qquad \text{for all } (x,y) \in D \tag{5.4}$$

The **zero** BCs in (5.3) lead to

(5.5)

(5.6)

Step 2: Solve X–problem. (Important: **two zero BCs!**)

It follows from (5.4) that

$$-X''(x) = \lambda X(x) \tag{5.7}$$

$X(x)$ also satisfies the Dirichlet conditions (5.5).

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n \in \mathbb{N} \tag{5.8}$$

Step 3: Solve Y–problem. (Important: λ_n **known from Step 2!**)

Let $n \in \mathbb{N}$. It follows from (5.4) with $\lambda = \lambda_n$ that

$$Y_n''(y) = \lambda_n Y_n(y) \tag{5.9}$$

$Y_n(y)$ also satisfies (5.6), i.e. $Y_n(0) = 0$.

and so

$$Y_n(y) \;=\; A_n\left(e^{\sqrt{\lambda_n}y} - e^{-\sqrt{\lambda_n}y}\right) \;=\; A_n \sinh\left(\sqrt{\lambda_n}y\right) \tag{5.10}$$

Step 4: Build up general solution.

$$u(x,y) \;=\; \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \tag{5.11}$$

Step 5: Satisfy non-zero BC $u(x,b) \;=\; g(x)$**.**

$$u(x,b) \;=\; \sum_{n=1}^{\infty}\left[A_n \sinh\left(\frac{n\pi b}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right)\,, \qquad 0 < x < a \tag{5.12}$$

Develop $g(x)$ in a **half-range Fourier sine series on** $(0,a)$:

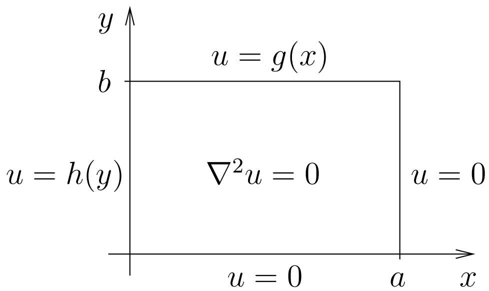
$$\tag{5.13}$$

and so

$$u(x,y) \;=\; \sum_{n=1}^{\infty}\left[\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a g(\tilde{x})\sin\left(\frac{n\pi\tilde{x}}{a}\right)\,d\tilde{x}\right]\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

is the solution of **BVP1**.

BVP2: Find a solution $u : D \to \mathbb{R}$ of (5.2) on D which satisfies the **(Dirichlet) boundary conditions**:



$$\left.\begin{array}{rcl} u(0,y) &=& h(y) \\ u(a,y) &=& 0 \end{array}\right\} \text{ for } \quad 0<y<b \qquad \left.\begin{array}{rcl} u(x,0) &=& 0 \\ u(x,b) &=& g(x) \end{array}\right\} \text{ for } \quad 0<x<a \tag{5.14}$$

where $g : (0,a) \to \mathbb{R}$ is a given function of x and $h : (0,b) \to \mathbb{R}$ is a given function of y.

Use the **superposition principle** (Theorem 5.4):

Find $v : D \to \mathbb{R}$ s.t. $\nabla^2 v = 0$ on D and	Find $w : D \to \mathbb{R}$ s.t. $\nabla^2 w = 0$ on D and
$\left.\begin{array}{rcl} v(0,y) &=& 0 \\ v(a,y) &=& 0 \end{array}\right\}$ for $0<y<b$	$\left.\begin{array}{rcl} w(0,y) &=& h(y) \\ w(a,y) &=& 0 \end{array}\right\}$ for $0<y<b$
$\left.\begin{array}{rcl} v(x,0) &=& 0 \\ v(x,b) &=& g(x) \end{array}\right\}$ for $0<x<a$	$\left.\begin{array}{rcl} w(x,0) &=& 0 \\ w(x,b) &=& 0 \end{array}\right\}$ for $0<x<a$

Then $u := v + w$ satisfies $\nabla^2 u = 0$ by Theorem 5.4 and

Solve v–problem $\iff$ **BVP1**.

Hence

$$v(x,y) = \sum_{n=1}^{\infty} \left[\frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a g(\tilde{x}) \sin\left(\frac{n\pi\tilde{x}}{a}\right) d\tilde{x} \right] \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) . \tag{5.15}$$

Solve w–problem.

Step 0: Substitute $\tilde{x} := a - x$ and $\tilde{w}(\tilde{x}, y) := w(x,y)$.

Obviously $\nabla^2 \tilde{w} = 0$ and

This is **BVP1** with the roles of x and y and a and b interchanged. Hence

$$\tilde{w}(\tilde{x}, y) \;=\; \sum_{n=1}^{\infty} \left[\frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b h(\tilde{y}) \sin\left(\frac{n\pi \tilde{y}}{b}\right) d\tilde{y} \right] \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi \tilde{x}}{b}\right) .$$

Resubstituting we get

$$w(x, y) \;=\; \sum_{n=1}^{\infty} \left[\frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b h(\tilde{y}) \sin\left(\frac{n\pi \tilde{y}}{b}\right) d\tilde{y} \right] \sinh\left(\frac{n\pi (a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad (5.16)$$

and adding (5.15) and (5.16) we get the solution $u(x, y)$ of **BVP2**.

Remark 5.6. (a) Obviously this procedure can be extended to the case of three/four non-zero (Dirichlet) boundary conditions by superposing the solutions of three/four problems with only one non-zero BC.

(b) For Neumann boundary conditions use the solutions to Case (b) of the eigenproblem in Section 4.5.

5.2.2 Circular Geometry

Consider now the annulus $D := \{(x, y) : 1 < x^2 + y^2 < b^2\}$. Using planar polar coordinates, i.e. $R \ldots$ distance from origin, $\phi \ldots$ polar angle, we get

$$D \;:=\; \{(R, \phi) : 1 < R < b,\; 0 \le \phi < 2\pi\} \qquad \text{and} \qquad u \;\equiv\; u(R, \phi). \qquad (5.17)$$

(**Note.** This is the same as using cylindrical polar coordinates with u independent of z.)

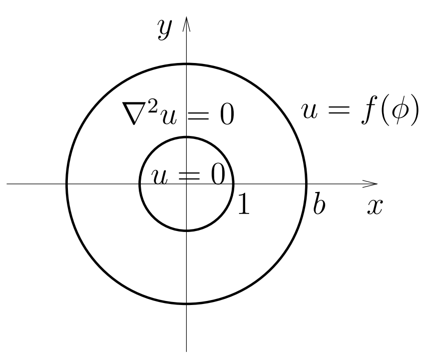
From Proposition 2.22(d) (see also *Problem Sheet 6, Question 4*), the Laplacian in **cylindrical polar coordinates** is

$$\nabla^2 u \;=\; \frac{1}{R} \left[\frac{\partial}{\partial R}\left(R \frac{\partial u}{\partial R}\right) \;+\; \frac{\partial}{\partial \phi}\left(\frac{1}{R}\frac{\partial u}{\partial \phi}\right) \;+\; \frac{\partial}{\partial z}\left(R \frac{\partial u}{\partial z}\right) \right]$$

Since $u = u(R, \phi)$ does not depend on z (in 2D-case):

$$\nabla^2 u \;=\; \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial u}{\partial R}\right) \;+\; \frac{1}{R^2}\frac{\partial^2 u}{\partial \phi^2} \qquad (5.18)$$

BVP3: Find a solution $u : D \to \mathbb{R}$ of (5.2) on D which satisfies **(Dirichlet) boundary conditions** for R and **periodic boundary conditions** for ϕ, i.e.



$$\left.\begin{aligned} \left.\begin{aligned} u(1, \phi) &= 0 \\ u(b, \phi) &= f(\phi) \end{aligned}\right\} &\quad \text{for} \quad 0 \le \phi < 2\pi \\ \left.\begin{aligned} u(R, 0) &= u(R, 2\pi) \\ \frac{\partial u}{\partial \phi}(R, 0) &= \frac{\partial u}{\partial \phi}(R, 2\pi) \end{aligned}\right\} &\quad \text{for} \quad 1 < R < b \end{aligned}\right\} \qquad (5.19)$$

where $f(\phi) = \sin^2 \phi$.

Step 1: **Separation of variables.**

Assume that $u(R,\phi) = F(R)\,G(\phi)$. Then using (5.18) $\nabla^2 u = 0$ is equivalent to

$$\frac{1}{R}\,(RF'G)' + \frac{1}{R^2}FG'' \;=\; \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 u}{\partial \phi^2} \;=\; \nabla^2 u \;=\; 0$$

$$\Leftrightarrow \qquad GR(RF')' + FG'' \;=\; 0$$

$$\Leftrightarrow \qquad \frac{R^2F''(R) + RF'(R)}{F(R)} \;=\; -\frac{G''(\phi)}{G(\phi)} \;=\; \lambda\ (= const), \qquad \text{for all } (R,\phi) \in D. \tag{5.20}$$

The **zero** & **periodic** boundary conditions in (5.19) lead to

$$0 \;=\; u(1,\phi) \;=\; F(1)\,G(\phi), \qquad 0 \le \phi < 2\pi \qquad \Leftrightarrow \qquad F(1) \;=\; 0 \tag{5.21}$$

$$\left.\begin{array}{rcl} F(R)\,G(0) &=& F(R)\,G(2\pi), \\ F(R)\,G'(0) &=& F(R)\,G'(2\pi), \end{array}\right\} \quad 1 < R < b \qquad \Leftrightarrow \qquad \left\{\begin{array}{rcl} G(0) &=& G(2\pi), \\ G'(0) &=& G'(2\pi). \end{array}\right\} \tag{5.22}$$

Step 2: **Solve the G-problem.**

$$-G''(\phi) \;=\; \lambda G(\phi), \qquad \text{for all } \ 0 \le \phi < 2\pi. \tag{5.23}$$

This is an eigenproblem of the form studied in Section 4.5 with periodic boundary conditions (5.22), i.e. equivalent to **Case (d)** in Section 4.5 (see also *Problem Sheet 10, Question 1(ii)*). Equation (5.23) has general solution

$$G(\phi) = A\sin\left(\sqrt{\lambda}\phi\right) + B\cos\left(\sqrt{\lambda}\phi\right)$$

again. However, using the boundary conditions (5.22) we have

$$\left.\begin{array}{rcl} B &=& A\sin 0 + B\cos 0 \;=\; A\sin\left(2\pi\sqrt{\lambda}\right) + B\cos\left(2\pi\sqrt{\lambda}\right) \\ A\sqrt{\lambda} &=& A\sqrt{\lambda}\,\cos 0 - B\sqrt{\lambda}\,\sin 0 \;=\; A\sqrt{\lambda}\,\cos\left(2\pi\sqrt{\lambda}\right) - B\sqrt{\lambda}\,\sin\left(2\pi\sqrt{\lambda}\right) \end{array}\right\}$$

$$\Leftrightarrow \qquad \begin{pmatrix} \sin(2\pi\sqrt{\lambda}) & \left(\cos(2\pi\sqrt{\lambda}) - 1\right) \\ \sqrt{\lambda}\left(1 - \cos(2\pi\sqrt{\lambda})\right) & \sqrt{\lambda}\,\sin(2\pi\sqrt{\lambda}) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.24}$$

The linear system (5.24) has a non-trivial solution $(A,B) \neq (0,0)$ only if the matrix has zero determinant, i.e.

$$\sqrt{\lambda}\,\sin^2(2\pi\sqrt{\lambda}) \;+\; \sqrt{\lambda}\left(\cos(2\pi\sqrt{\lambda}) - 1\right)^2 \;=\; 0$$

$$\Leftrightarrow \qquad \sqrt{\lambda}\left(\sin^2(2\pi\sqrt{\lambda}) \;+\; \cos^2(2\pi\sqrt{\lambda}) \;-\; 2\cos(2\pi\sqrt{\lambda}) \;+\; 1\right) \;=\; 0$$

$$\Leftrightarrow \qquad \sqrt{\lambda}\left(1 \;-\; \cos(2\pi\sqrt{\lambda})\right) \;=\; 0$$

Therefore, either $\lambda = 0$ or $\cos(2\pi\sqrt{\lambda}) = 1$ (i.e. $\sqrt{\lambda} = n,\ n \in \mathbb{Z}$) and the solutions to the G-problem are

$$\left.\begin{aligned} \lambda_n &= n^2, \\ G_n(\phi) &= A_n \cos(n\phi) + B_n \sin(n\phi)\,, \end{aligned}\right\} \quad n \in \mathbb{N} \cup \{0\}. \tag{5.25}$$

Step 3: Solve the F-problem. (with λ_n given)

Case $n = 0$: $\quad R^2 F_0''(R) + R F_0'(R) = 0, \quad 1 < R < b,$

which has general solution $F_0(R) = C_0 \ln R + D_0$. However, it follows from (5.21) that $D_0 = F_0(1) = 0$ and so

$$F_0(R) = C_0 \ln R. \tag{5.26}$$

Case $n \geq 1$: $\quad R^2 F_n''(R) + R F_n'(R) - n^2 F_n(R) = 0, \quad 1 < R < b,$

Try R^μ:

$$\begin{aligned} R^2\,\mu(\mu-1)R^{\mu-2} + R\,\mu R^{\mu-1} - n^2 R^\mu &= 0 \\ \Leftrightarrow \qquad R^\mu(\mu^2 - \mu + \mu - n^2) &= 0 \qquad \Leftrightarrow \qquad \mu = \pm n. \end{aligned}$$

Hence, $F_n(R) = C_n R^n + D_n R^{-n}$. However, it follows from (5.21) again that $C_n + D_n = F_n(1) = 0$ and therefore

$$F_n(R) = C_n \left(R^n - R^{-n}\right). \tag{5.27}$$

Step 4: Building up the general solution.

Putting Steps 2 & 3 together and using the superposition principle (Theorem 5.4) we get

$$u(R,\phi) = \tilde{A}_0 \ln R + \sum_{n=1}^{\infty} \left(R^n - R^{-n}\right)\left(\tilde{A}_n \cos(n\phi) + \tilde{B}_n \sin(n\phi)\right) \tag{5.28}$$

for all $(R,\phi) \in D$ (where $\tilde{A}_n := C_n A_n$ and $\tilde{B}_n := C_n B_n$).

Step 5: Satisfy the non-zero boundary condition $u(b,\phi) = \sin^2\phi$**.**

$$u(b,\phi) = [\tilde{A}_0 \ln b] + \sum_{n=1}^{\infty} \left[\tilde{A}_n(b^n - b^{-n})\right] \cos(n\phi) + \left[\tilde{B}_n(b^n - b^{-n})\right] \sin(n\phi)\,,$$

but

$$u(b,\phi) = \sin^2\phi = \frac{1}{2}(1 - \cos 2\phi)$$

and so by comparing coefficients we get

$$\tilde{A}_0 = \frac{1}{2\ln b}\,, \qquad \tilde{A}_2 = -\frac{1}{2(b^2 - b^{-2})}\,, \qquad \tilde{A}_n = \tilde{B}_n = 0 \quad \text{otherwise.}$$

Therefore the solution of **BVP3** is

$$u(R,\phi) = \frac{\ln(R-b)}{2} - \frac{R^2 - R^{-2}}{2(b^2 - b^{-2})} \cos 2\phi\,.$$

5.2.3 Hyperbolic PDEs – Wave Equation

Let us now look at the **wave equation**

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{5.29}$$

where c is a given constant and let

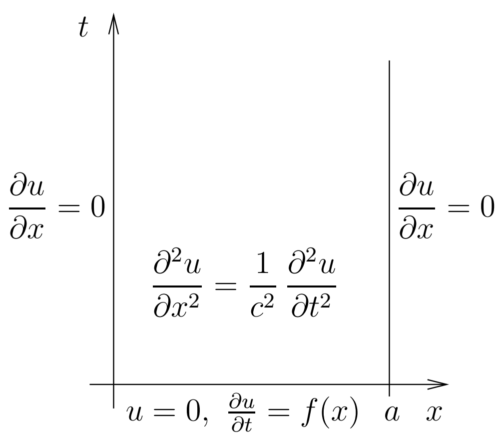
$$D := \{(x,t) \in \mathbb{R}^2 : 0 < x < a,\ t > 0\}.$$

Equation (5.29) takes the general form (5.1) with $a = 1$, $b = -1/c^2$ and $h = 0$. Therefore $ab - h^2 = -1/c^2 < 0$ and (5.29) is **hyperbolic**.

Applications. Wave propagation (e.g. vibrating string, water waves, electromagnetic waves)

IMPORTANT. Usually x is the **spatial variable** and t is the **time variable**, and the choice of initial/boundary conditions is very important (not uniquely solvable for every choice!)

IBVP1: Find a solution $u : D \to \mathbb{R}$ of (5.29) on D which satisfies the **initial conditions**



$$\left.\begin{aligned} u(x,0) &= 0 \\ \frac{\partial u}{\partial t}(x,0) &= f(x) \end{aligned}\right\} \text{ for } \quad 0 < x < a \tag{5.30}$$

and the **(Neumann) boundary conditions**

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(a,t) &= 0 \end{aligned}\right\} \text{ for } \quad t > 0 \tag{5.31}$$

where $f : (0,a) \to \mathbb{R}$ is a given function of x.

Step 1: Separation of variables.

Assume that $u(x,y) = X(x)T(t))$ is a solution of (5.29). Then (5.29) becomes

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

Therefore

$$-\frac{X''(x)}{X(x)} = -\frac{1}{c^2}\frac{T''(t)}{T(t)} = \lambda \quad (= const), \qquad \text{for all } (x,t) \in D. \tag{5.32}$$

The **zero** initial and boundary conditions in (5.30) and (5.31) lead to

$$0 = u(x,0) = X(x)T(0), \qquad 0 < x < a \qquad \Leftrightarrow \qquad T(0) = 0 \tag{5.33}$$

$$\left.\begin{aligned} 0 &= \frac{\partial u}{\partial x}(0,t) = X'(0)T(t), \\ 0 &= \frac{\partial u}{\partial x}(a,t) = X'(a)T(t), \end{aligned}\right\} \quad t > 0 \qquad \Leftrightarrow \qquad \left\{\begin{aligned} X'(0) &= 0, \\ X'(a) &= 0. \end{aligned}\right. \tag{5.34}$$

Step 2: **Solve X–problem.**

It follows from (5.32) that

$$-X''(x) \;=\; \lambda X(x) \tag{5.35}$$

$X(x)$ also satisfies the Neumann conditions (5.34). This is Case (b) of the eigenproblem in Section 4.5 and has solutions

$$\lambda_n \;=\; \frac{n^2\pi^2}{a^2}\,, \qquad X_n(x) \;=\; \cos\left(\frac{n\pi x}{a}\right)\,, n\in\mathbb{N}\cup\{0\}. \tag{5.36}$$

Step 3: **Solve T–problem.** (with λ_n from Step 2)

Case $n\in\mathbb{N}$. It follows from (5.32) with $\lambda=\lambda_n$ that

$$-T_n''(t) \;=\; \lambda_n c^2 T_n(t) \tag{5.37}$$

which has general solution

$$T_n(t) \;=\; A_n\sin(\sqrt{\lambda_n}ct) \;+\; B_n\cos(\sqrt{\lambda_n}ct).$$

However, $T_n(t)$ also satisfies (5.33) and so

$$0 \;=\; T_n(0) \;=\; A_n\sin 0 \;+\; B_n\cos 0 \;=\; B_n$$

and

$$T_n(t) \;=\; A_n\sin\left(\frac{n\pi ct}{a}\right)\,. \tag{5.38}$$

Case $n=0$. In this case we have again from (5.32) that

$$-T_0''(t) \;=\; 0$$

which has general solution $T_0(t)=A_0t+B_0$ but since $0=T_0(0)=B_0$ we get

$$T_0(t) \;=\; A_0 t\,. \tag{5.39}$$

Step 4: **Build up general solution.**

Putting Steps 2 and 3 together and using the superposition principle (Theorem 5.4) we get

$$u(x,t) \;=\; A_0t \;+\; \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right) \tag{5.40}$$

Step 5: **Satisfy non-zero BC** $\dfrac{\partial u}{\partial t}(x,0) \;=\; f(x)$**.**

$$\frac{\partial u}{\partial t}(x,0) \;=\; A_0 \;+\; \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{a}\right)\frac{n\pi c}{a}\cos 0 \;=\; A_0+\sum_{n=1}^{\infty}\left[\frac{n\pi c A_n}{a}\right]\cos\left(\frac{n\pi x}{a}\right) \tag{5.41}$$

for $0<x<a$. Develop $f(x)$ in a **half-range Fourier cosine series on** $(0,a)$:

$$f(x) \;=\; \left[\frac{1}{a}\int_0^a f(\tilde{x})\,d\tilde{x}\right] \;+\; \sum_{n=1}^{\infty}\left[\frac{2}{a}\int_0^a f(\tilde{x})\,\cos\left(\frac{n\pi\tilde{x}}{a}\right)d\tilde{x}\right]\cos\left(\frac{n\pi x}{a}\right). \tag{5.42}$$

Comparing coefficients in (5.41) and (5.42) we finally get that the coefficients of the solution to **IBVP1** in (5.40) are

$$A_0 \;=\; \frac{1}{a}\int_0^a f(\tilde{x})\,d\tilde{x}, \qquad \text{and} \quad A_n \;=\; \frac{2}{n\pi c}\int_0^a f(\tilde{x})\cos\left(\frac{n\pi\tilde{x}}{a}\right)d\tilde{x}.$$

5.2.4 Parabolic PDEs – Diffusion Equation

Finally for an example of a parabolic PDE let us look at the **diffusion equation**

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{5.43}$$

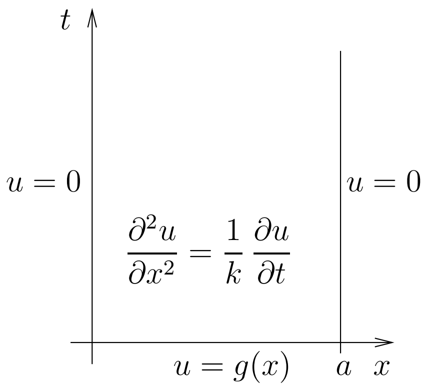
where again k is a given constant and

$$D := \{(x,t) \in \mathbb{R}^2 : 0 < x < a,\ t > 0\}.$$

Equation (5.43) takes the general form (5.1) with $a = 1$ and $b = h = 0$. Therefore $ab - h^2 = 0$ and (5.43) is **parabolic**.

Applications. Flow of heat (unsteady!), flow in porous media (e.g. groundwater flow), permeation of gases in polymers, etc.

IBVP2: Find a solution $u : D \to \mathbb{R}$ of (5.43) on D which satisfies the **initial condition**



$$u(x,0) = g(x), \qquad 0 < x < a \tag{5.44}$$

and the **(Dirichlet) boundary conditions**

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(a,t) &= 0 \end{aligned}\right\} \text{ for } \quad t > 0 \tag{5.45}$$

where $g(x) = x^2$.

See *Problem Sheet 10, Question 3.*

5.3 Other Solution Methods for PDEs (not examinable!)

5.3.1 The Laplace Transform Method

Question: In **IBVP1** and **IBVP2**, what can we do when the boundary conditions are not homogeneous? E.g. in **IBVP2** let

$$u(x,0) = 0, \qquad \text{for} \quad 0 < x < a,$$

$$\begin{aligned} u(0,t) &= 0, \\ u(a,t) &= U, \qquad \text{for} \quad t > 0. \end{aligned}$$

Answer: We can take a Laplace transform with respect to time (**recall from MA20009**).

Example 5.7. (Diffusion Equation). Let

$$D := \{(x,t) \in \mathbb{R}^2 : 0 < x < a,\ t > 0\}.$$

IBVP3: Find a solution $u : D \to \mathbb{R}$ of the diffusion equation (5.43) on D which satisfies the initial condition

$$u(x,0) \;=\; 0, \qquad 0 < x < a \tag{5.46}$$

and the boundary conditions

$$\left.\begin{array}{rcl} u(0,t) & = & 0 \\ u(a,t) & = & U \end{array}\right\} \text{ for } \quad t > 0 \tag{5.47}$$

where U is constant.

Let $\hat{u}$ denote the Laplace transform of u with respect to t (for fixed x), i.e.

$$\hat{u}(x,s) \;:=\; \mathcal{L}\Big\{t \mapsto u(x,t)\Big\}(s) \;=\; \int_0^\infty u(x,t)e^{-st}dt.$$

Step 1: Apply Laplace transform w.r.t. time to (5.43)

$$\begin{aligned} \int_0^\infty \frac{\partial^2 u}{\partial x^2}(x,t)e^{-st}\,dt \;&=\; \int_0^\infty \frac{1}{k}\,\frac{\partial u}{\partial t}(x,t)e^{-st}\,dt \\ \Leftrightarrow \qquad \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)e^{-st}\,dt \;&=\; \frac{1}{k}\left[\Big(u(x,t)\,e^{-st}\Big|_{t=0}^{\infty} + s\int_0^\infty u(x,t)e^{-st}\,dt\right] \end{aligned}$$

Therefore, using the initial condition (5.46) we get

$$\frac{\partial^2 \hat{u}}{\partial x^2}(x,s) \;=\; \frac{s}{k}\,\hat{u}(x,s). \tag{5.48}$$

[So rather than an algebraic equation (as in **MA20009**) we get an ODE in x.]

Step 2: Apply Laplace transform to the BCs (5.47)

$$\left.\begin{array}{rcl} \hat{u}(0,s) & = & \displaystyle\int_0^\infty u(0,t)e^{-st}\,dt \;=\; 0, \\ \hat{u}(a,s) & = & \displaystyle\int_0^\infty u(a,t)e^{-st}\,dt \;=\; \int_0^\infty U\,e^{-st}\,dt \;=\; \frac{U}{s}\,. \end{array}\right\} \tag{5.49}$$

Step 3: Solve the ODE (5.48) & (5.49)

The general solution of (5.48) is

$$\hat{u}(x,s) \;=\; Ae^{\sqrt{s/k}\,x} + Be^{-\sqrt{s/k}\,x}$$

However, it follows from (5.49) that

$$0 \;=\; \hat{u}(0,s) \;=\; A + B \quad\Rightarrow\quad B \;=\; -A \quad\Rightarrow\quad \hat{u}(x,s) \;=\; 2A\sinh\left(\sqrt{\frac{s}{k}}\;x\right)$$

and

$$\frac{U}{s} \;=\; \hat{u}(a,s) \;=\; 2A\sinh\left(\sqrt{\frac{s}{k}}a\right)\;.$$

Hence,

$$\hat{u}(x,s) \;=\; \frac{U\sinh(\sqrt{s/k}\;x)}{s\sinh(\sqrt{s/k}\;a)}\,. \tag{5.50}$$

Step 4: Invert the Laplace transform (not always easy!)

Finally, using the inverse transformation or a Laplace transform table (e.g. see [Constanda, pp. 167]) we get

$$u(x,t) \;=\; U\left(\frac{x}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\;\sin\left(\frac{n\pi x}{a}\right)\;e^{-n^2\pi^2kt/a^2}\right). \tag{5.51}$$

Remark 5.8. A related method to the Laplace Transform Method is the Fourier Transform Method. Fourier transforms can be used for example for the diffusion equation to eliminate space dependency, if the partial differential equation is posed on the entire real line $x\in\mathbb{R}$. For more details see e.g. [Constanda, Chapter 8] or [**MA30044**].

5.3.2 D'Alembert's Solution of the Wave Equation

We will finish with the solution for the wave equation posed on the entire real line with sufficiently smooth initial data. This problem can be solved explicitly and the solution is called **D'Alembert's solution of the wave equation** (see [**MA30044**] for details):

IBVP4: Find $u:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ such that $u(x,t)$ satisfies the **wave equation** (5.29) and the initial conditions

$$\left.\begin{aligned} u(x,0) &= \phi(x) \\ \frac{\partial u}{\partial t}(x,0) &= \psi(x) \end{aligned}\right\}\quad \text{for all}\quad x\in\mathbb{R}, \tag{5.52}$$

where ϕ is twice continuously differentiable and ψ is once continuously differentiable.

First we show that any solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} \;=\; \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2}$$

can be written as

$$u(x,t) \;=\; f(x-ct)\;+\;g(x+ct) \tag{5.53}$$

where f and g are arbitrary twice continuously differentiable functions. To do this let us introduce

$$\xi \;=\; x\;-\;ct \qquad \text{and} \qquad \eta \;=\; x\;+\;ct$$

and

$$U(\xi,\eta) \;=\; u(x,t)\,.$$

It is an easy exercise to verify that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 U}{\partial \xi^2} \;+\; 2\frac{\partial^2 U}{\partial\xi\,\partial\eta} \;+\; \frac{\partial^2 U}{\partial \eta^2} \qquad \text{and} \\ \frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 U}{\partial \xi^2} \;-\; 2\frac{\partial^2 U}{\partial\xi\,\partial\eta} \;+\; \frac{\partial^2 U}{\partial \eta^2}\right). \end{aligned}$$

Hence the wave equation (5.29) reduces to

$$\frac{\partial^2 U}{\partial\xi\,\partial\eta} = 0 .$$

Integrating with respect to ξ we get

$$\frac{\partial U}{\partial\eta} = g'(\eta),$$

i.e. the derivative of an arbitrary function of η. Now integrating with respect to η we get

$$U(\xi,\eta) = g(\eta) + f(\xi)$$

where f is an arbitrary function of ξ. Resubstituting ξ and η we have established (5.53).

Now using the initial conditions (5.52) we have

$$\begin{aligned} g(x) + f(x) &= \phi(x) \\ cg'(x) - cf'(x) &= \psi(x) \end{aligned}$$

Integrating the last equation from and arbitrary point $x_0 \in \mathbb{R}$ to x we get

$$g(x) - f(x) = \frac{1}{c}\int_{x_0}^{x} \psi(\xi)d\xi .$$

Hence we can solve for $f(x)$ and $g(x)$ to obtain

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int_{x_0}^{x}\psi(\xi)d\xi\right) \qquad \text{and} \qquad g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int_{x_0}^{x}\psi(\xi)d\xi\right)$$

Finally, substituting back into (5.53) we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\phi(x-ct) - \frac{1}{c}\int_{x_0}^{x-ct}\psi(\xi)d\xi + \phi(x+ct) + \frac{1}{c}\int_{x_0}^{x+ct}\psi(\xi)d\xi\right) \\ &= \frac{\phi(x-ct) + \phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi \qquad (5.54) \end{aligned}$$

which is the so-called **D'Alembert solution of the wave equation**.