Chapter 4

Fourier Series [Constanda, pp. 11–27]

Motivation. Suppose f is a smooth function (all derivatives exist). Set

$$f(x) \;=\; a_0 \;+\; a_1 x \;+\; a_2 x^2 \;+\; \dots$$

Therefore

$$f(x) \;=\; f(0) \;+\; \sum_{n=1}^{\infty} \frac{1}{n!}\, f^{(n)}(0)\, x^n \qquad \text{(McLaurin series)}$$

Instead of expanding $f(x)$ as a polynomial we now expand it as a **trigonometric polynomial**.

Definition 4.1. Let $L > 0$. A continuous function $f : (-L, L) \to \mathbb{R}$ has a **Fourier series**, if there are coefficients $\{a_n\}$ and $\{b_n\}$ such that $f(x)$ can be written as

$$f(x) \;=\; \underbrace{a_0 \;+\; \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) \;+\; b_n \sin\left(\frac{n\pi x}{L}\right) \right]}_{=:\, S(x)} \qquad \text{for all } x \in (-L, L) \tag{4.1}$$

Lemma 4.2 (orthogonality relations). *Let $m, n \in \mathbb{N}_0$. The following identities hold:*

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\, \cos\left(\frac{n\pi x}{L}\right)\, dx \;=\; \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases} \tag{4.2}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\, \sin\left(\frac{n\pi x}{L}\right)\, dx \;=\; \begin{cases} 0 & \text{if } m \neq n, \text{ or } m = n = 0 \\ L & \text{if } m = n \neq 0 \end{cases} \tag{4.3}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\, \cos\left(\frac{n\pi x}{L}\right)\, dx \;=\; 0 \tag{4.4}$$

Proof.

$$I \; = \; \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \, \cos\left(\frac{n\pi x}{L}\right) \, dx \; =$$

$\square$

Proposition 4.3. *The Fourier coefficients in (4.1) are* $a_0 \; = \; \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x)\, dx,$

$$a_n \; = \; \frac{1}{L} \int_{-L}^{L} f(x) \, \cos\left(\frac{n\pi x}{L}\right) \, dx \qquad \text{and} \qquad b_n \; = \; \frac{1}{L} \int_{-L}^{L} f(x) \, \sin\left(\frac{n\pi x}{L}\right) \, dx \tag{4.5}$$

Proof. $\boxed{a_0\textbf{:}}$ Integrate (4.1) over $(-L, L)$. Then

$\boxed{a_m, \;\; m \geq 1\textbf{:}}$ Multiply (4.1) by $\cos\left(\dfrac{n\pi x}{L}\right)$, integrate over $(-L, L)$ and then use Lemma 4.2.

$\boxed{b_m, \;\; m \geq 1\textbf{:}}$ Multiply (4.1) by $\sin\left(\dfrac{n\pi x}{L}\right)$, integrate over $(-L, L)$ and use Lemma 4.2.

$\square$

Example 4.4. $f(x) \;=\; x^2,\;\; x \in (-1,1) \quad$ (i.e. $L = 1$).

$$a_0 \;=$$

To calculate a_n use integration by parts (twice):

Similarly,

$$b_n \;=$$

Therefore

$$x^2 \;=\; \frac{1}{3} \;+\; \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\;\cos n\pi x\;, \qquad\qquad x \in (-1,1).$$

By fixing x we can derive some useful results.

(a) $x = 0$:

(b) $x = 1$:

Example 4.5. $f(x) \;=\; x,\;\; x \in (-1,1) \quad$ (i.e. $L = 1$).

$$a_0 \;=$$

$$a_n \;=$$

$$b_n \;=$$

Therefore

$$x \;=\; \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\;\sin n\pi x\;, \qquad\qquad x\in(-1,1).$$

4.1 Periodicity of Fourier Series

In Definition 4.1 we have defined the Fourier series

$$S(x) \;:=\; a_0 \;+\; \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) \;+\; b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

for $x\in(-L,L)$. Let S (above) also be defined at $x=L$. We can extend $S(x)$ to all $x\in\mathbb{R}$.

Proposition 4.6.

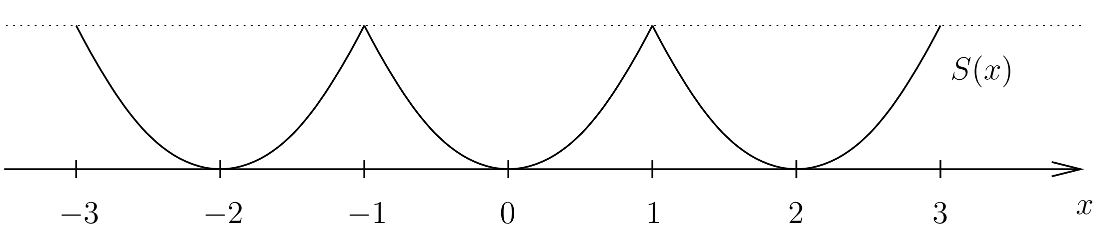
$$S(x+2L) \;=\; S(x) \qquad\qquad \text{for all } x\in\mathbb{R}. \tag{4.6}$$

Proof.

□

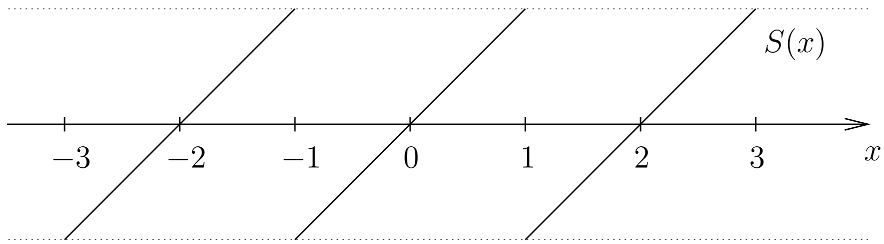
Hence, $S:\mathbb{R}\to\mathbb{R}$ is a $2L$-periodic function that coincides with f on (-L,L).

Recall Example 4.4:



4.2 Fourier Convergence

So far we have only looked at Fourier series for continuous functions. Let us now look at the general Fourier convergence theory.
Recall Example 4.5:



Note. Although f is continuous, the Fourier series is discontinuous. To see this, let $\varepsilon > 0$.

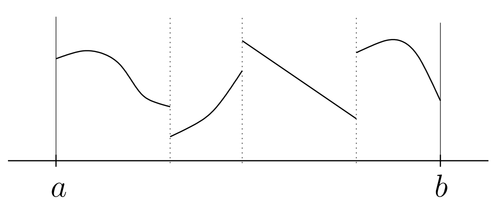
The Fourier series is **discontinuous** either when

(a) f has a discontinuity in (-L,L), or

(b) $f(-L) \neq f(L)$.

Definition 4.7. Let $\varepsilon > 0$.

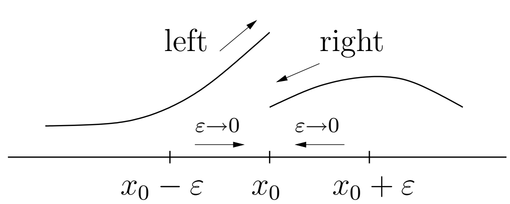
(a) A bounded function $f : (a, b) \to \mathbb{R}$ is **piecewise continuous** on (a, b), if we can subdivide (a, b) into finitely many (sub)intervals in each of which f is continuous, e.g.



(b) [**MA20007, to come**] The **left-** and **right-hand limit** of f at x_0 are defined as

$$f(x_0 - 0) := \lim_{\varepsilon \to 0} f(x_0 - \varepsilon) \qquad \text{and} \qquad f(x_0 + 0) := \lim_{\varepsilon \to 0} f(x_0 + \varepsilon) \ ,$$

respectively.



Note. If f is continuous at x_0, then $f(x_0 - 0) = f(x_0 + 0) = f(x_0)$.

(c) The **left-** and **right-hand derivatives** of f at x_0 are defined as

$$\lim_{\varepsilon \to 0} \frac{f(x_0 - 0) - f(x_0 - \varepsilon)}{\varepsilon} \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon) - f(x_0 + 0)}{\varepsilon},$$

respectively.

Theorem 4.8. *Suppose $f : (-L, L) \to \mathbb{R}$ is bounded and piecewise continuous on $(-L, L)$, and suppose the left- and right-hand derivatives of f exist for all $x_0 \in [-L, L]$. Then f has a Fourier series $S(x)$ and*

$$S(x) = \begin{cases} \frac{1}{2}\Big[f(x-0) + f(x+0)\Big], & \text{if } x \in (-L, L), \\ \frac{1}{2}\Big[f(L-0) + f(-L+0)\Big], & \text{if } x = L \text{ or } x = -L. \end{cases} \tag{4.7}$$

Proof. See for example [Churchill, R.V., "Fourier Series and Boundary Value Problems"]. □

4.3 Gibbs' Phenomenon

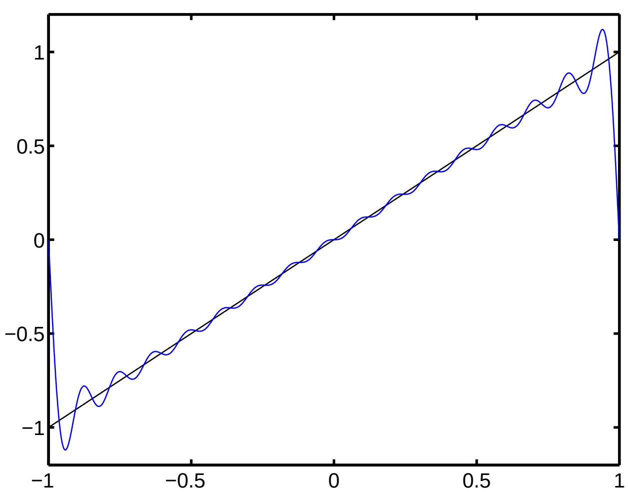
Let us look at the partial sums

$$S_N(x) := a_0 + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{4.8}$$

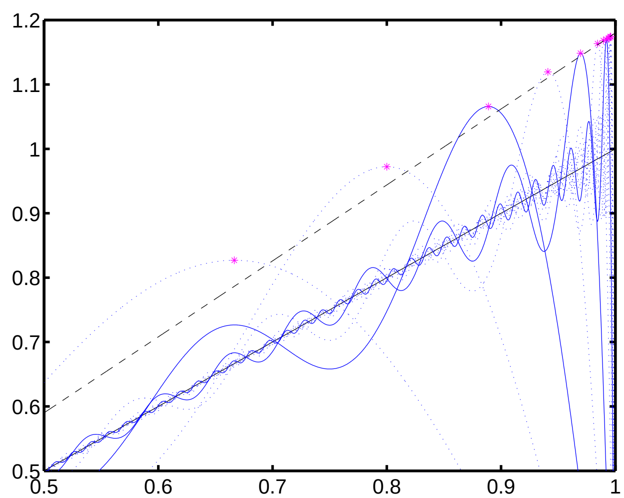
for Examples 4.4 and 4.5.

Comparing the plots of $S_N(x)$ for various N in both examples (see handout) we see that

- if $S(x)$ is continuous at L (e.g. Example 4.4), we have very fast convergence;
- if $S(x)$ is discontinuous at L (e.g. Example 4.5), we have very slow convergence and "overshoots" near the discontinuity. This is **"Gibbs' phenomenon"**:



$S_{16}(x)$ for Example 4.5.



$S_2(x)$ to $S_{256}(x)$ for Example 4.5.

In fact,

(i)

(ii)

This is **no contradiction**. Both (i) and (ii) can be true together, because x_N gets closer and closer to 1 as N increases, leaving a convergent region behind (see the figure above). This is called **non-uniform convergence**.

4.4 Half-range Series

Definition 4.9. (a) A function $f(x)$ is called **even**, if $f(-x) = f(x)$.

(b) A function $f(x)$ is called **odd**, if $f(-x) = -f(x)$.

Examples.

Proposition 4.10. *(a) If $f(x)$ is even, then $b_n = 0$, for all $n \geq 1$,*

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\,, \quad \text{for all } n \geq 0. \tag{4.9}$$

(b) If $f(x)$ is odd, then $a_n = 0$, for all $n \geq 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\,, \qquad \text{for all } n \geq 1. \tag{4.10}$$

Proof. [Problem Sheet 9, Question 4]. □

Definition 4.11. Let $f : (0, L) \to \mathbb{R}$. Over the **half-range** $(0, L)$ we can expand $f(x)$ in a

(a) **half-range Fourier cosine series:**

$$S_c(x) := A_0 + \sum_{i=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \qquad x \in \mathbb{R}, \tag{4.11}$$

where $A_0 := \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx$ and $A_n := \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

(b) **half-range Fourier sine series:**

$$S_s(x) \;:=\; \sum_{i=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \;, \qquad x \in \mathbb{R}, \tag{4.12}$$

where $B_n := \dfrac{2}{L}\displaystyle\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right) dx$.

Remark 4.12. $S_c(x)$ and $S_s(x)$ are both $2L$–periodic and coincide with $f(x)$ on $(0, L)$, **but** $S_c(x)$ is even and $S_s(x)$ is odd.

Example 4.13. Let $f(x) = 1 - x$ for $x \in (0, 1)$ (i.e. $L = 1$):

4.5 Application: Eigenproblems for 2nd-order ODEs [Constanda, pp. 29–47]

Consider the following eigenproblem (special case of a Sturm-Liouville problem):

Find $y : (0, L) \to \mathbb{C}$ and $\lambda \in \mathbb{C}$ such that

$$-y''(x) \;=\; \lambda\, y(x) \;, \qquad \text{for all } x \in (0, L) \tag{4.13}$$

and such that y satisfies

(a) homogeneous Dirichlet boundary conditions:

$$y(0) \;=\; y(L) \;= 0 \tag{4.14}$$

(b) homogeneous Neumann boundary conditions:

$$y'(0) \;=\; y'(L) \;= 0 \tag{4.15}$$

(c) mixed boundary conditions:

$$y(0) \;=\; 0 \qquad \text{and} \qquad y'(L) \;=\; 0$$

(d) periodic boundary conditions:

$$y(0) \;=\; y(L) \qquad \text{and} \qquad y'(0) \;=\; y'(L)$$

Case (a):

Therefore the non-trivial, linearly independent solutions of (4.13), (4.14) are

$$\left.\begin{aligned} y_n(x) &= \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n &= \frac{n^2\pi^2}{L^2} \end{aligned}\right\} \quad n \in \mathbb{N}. \tag{4.16}$$

The y_n are called **eigenfunctions** of (4.13), (4.14). The λ_n are called **eigenvalues**.

Note. For $n = 0$: $y_0(x) = 0$ (i.e. trivial),
$n < 0$: $y_n(x) = -y_{-n}(x)$ (i.e. linearly dependent),
and all the eigenvalues and eigenfunctions are real-valued.

Remark 4.14 *(link to Fourier series).*

(i) The **half-range Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L,$$

expresses f in terms of eigenfunctions of (4.13), (4.14).

(ii) It follows from Lemma 4.2 that the eigenfunctions $y_n(x)$, $n \in \mathbb{N}$, of (4.13), (4.14) are **orthogonal** on $(0, L)$, i.e. for $m, n \in \mathbb{N}$

$$\int_0^L y_m(x)\, y_n(x) dx = \begin{cases} L/2 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Case (b):

Therefore the non-trivial, linearly independent solutions of (4.13), (4.15) are

$$\left.\begin{aligned} y_n(x) &= \cos\left(\frac{n\pi x}{L}\right) \\ \lambda_n &= \frac{n^2\pi^2}{L^2} \end{aligned}\right\} \quad n \in \mathbb{N} \cup \{0\}, \tag{4.17}$$

i.e. the **half-range Fourier cosine series** on $(0, L)$ expresses a function in terms of eigenfunctions of (4.13), (4.15), and the eigenfunctions $y_n(x)$, $n \in \mathbb{N} \cup \{0\}$, of (4.13), (4.15) are **orthogonal** on $(0, L)$ again.

Cases (c) and (d): [Problem Sheet 9, Question 5].