Chapter 2

Vector Calculus

2.1 Directional Derivatives and Gradients


Definition 2.1. Let $f : \Omega \to \mathbb{R}$ be a continuously differentiable scalar field on a region $\Omega \subset \mathbb{R}^3$. Then

$$\text{grad } f \equiv \nabla f := \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \quad (2.1)$$

is the gradient of $f$ on $\Omega$, which is itself a vector field on $\Omega$.

Definition 2.2. Let $f : \Omega \to \mathbb{R}$ be a continuously differentiable scalar field on $\Omega \subset \mathbb{R}^3$ and let $\hat{a}$ be a unit vector in $\mathbb{R}^3$. Then

$$D\hat{a}f(x_0) := \lim_{h \to 0} \frac{f(x_0 + h\hat{a}) - f(x_0)}{h} \quad (2.2)$$

is the directional derivative of $f$ in the direction $\hat{a}$ at $x_0 \in \Omega$, i.e. the rate of change of $f$ in the direction of $\hat{a}$. Moreover, for any $a \in \mathbb{R}^3$ we define $Daf := D\hat{a}f$ where $\hat{a} = a/|a|$.

Proposition 2.3. Let $a \in \mathbb{R}^3$. Then

$$Daf = \nabla f \cdot \frac{a}{|a|}. \quad (2.3)$$

Proof.
Example 2.4. Find $D_a f(x_0)$ for $f(x) := 2x^2 + 3y^2 + z^2$, $a := (1, 0, -2)^T$, and $x_0 := (2, 1, 3)^T$.

Step 1: Find $\nabla f$:

Step 2: Normalise $a$:

Step 3: Evaluate (2.3) at $x_0$:

Proposition 2.5.

$$\max_{a \in \mathbb{R}^3} |D_a f| = |\nabla f|$$

(2.6)

and it is attained in the direction $a = \nabla f$.

Proof.

Remark 2.6. (geometric interpretation of grad).

(a) Proposition 2.5 shows that $\nabla f(x_0)$ gives the direction and magnitude of the largest directional derivative of $f$ at $x_0$, i.e. the largest rate of change of $f$. For a picture illustrating this in 2D see [Anton, Figure 14.6.5, p. 978].

(b) A point $x_0 \in \Omega$ where $\nabla f(x_0) = 0$ is called a stationary point (since Proposition 2.5 shows that $D_a f(x_0) = 0$ for all $a \in \mathbb{R}^3$).

For work integrals the **Fundamental Theorem of Calculus** takes the following form.

Theorem 2.7. Let $\phi : \Omega \to \mathbb{R}$ be a continuously differentiable scalar field, and let $C$ be a curve in $\Omega$ from $x_0$ to $x_e$. Then

$$\int_C \nabla \phi \cdot dr = \phi(x_e) - \phi(x_0).$$

(2.7)

Proof. [Problem Sheet 2, Question 5].
2.1.1 Application: Level Surfaces and Grad

Note. Please do not confuse the definition of a level surface (as done in MA10005 or [Anton]) with our definition of a (parametric) surface in Equation (1.13).

2.2 Divergence and Curl; the $\nabla$–Operator

Let us consider $\nabla f$ again, but now think as $\nabla$ as an operator acting on a scalar field $f$. Formally we could write

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} = \begin{pmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{pmatrix}. \quad (2.8)$$

This is called the del-operator or nabla, and applying it to a scalar field $f$ we get

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} = \text{grad } f.$$  

Now let us apply $\nabla$ to vector fields. Let $F : \Omega \to \mathbb{R}^3$ be a vector field on a region $\Omega \subset \mathbb{R}^3$, where $F = F_1 i + F_2 j + F_3 k$. (As for ordinary vectors $a b$ does not make sense, but we can form the dot and the cross product, i.e. $a \cdot b$ and $a \land b$.)

**Definition 2.8.** We define the divergence of $F$ to be the scalar field

$$\text{div } F := \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad (2.9)$$

and the curl of $F$ to be the vector field

$$\text{curl } F := \nabla \land F = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) j + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k. \quad (2.10)$$

**Example 2.9.** Find div and curl of $F(x) = (-y, x, z)^T$.

**Remark 2.10.** In manipulating with the $\nabla$–operator many rules from ordinary vector algebra apply, but not all (don’t use vector algebra to prove them!). In particular, the order is very important, i.e. in general

$$\nabla \cdot (gF) \neq g\nabla \cdot F \neq F \cdot \nabla g.$$  

**Applications.** In many fields of mathematical physics, e.g. fluid flow, electromagnetic field propagation, etc...
2.2.1 Second order derivatives – the Laplace Operator

Applying the $\nabla$-operator twice gives five possible second derivatives:

- **A** \[ \text{div}(\text{grad} \, f) = \nabla \cdot (\nabla f) = \nabla^2 f \]  
  (2.11)

- **B** \[ \text{curl}(\text{grad} \, f) = \nabla \wedge (\nabla f) \]

- **C** \[ \text{div}(\text{curl} \, F) = \nabla \cdot (\nabla \wedge F) \]

- **D** \[ \text{curl}(\text{curl} \, F) = \nabla \wedge (\nabla \wedge F) \]

- **E** \[ \text{grad}(\text{div} \, F) = \nabla (\nabla \cdot F) \]

**Note.** Quantities such as \( \text{grad}(\text{curl} \, F) \) or \( \text{curl}(\text{div} \, F) \) are meaningless.

**Proposition 2.11.**

(a) \[ \text{curl}(\text{grad} \, f) = 0, \]
(b) \[ \text{div}(\text{curl} \, F) = 0, \]
(c) \[ \text{grad}(\text{div} \, F) - \text{curl}(\text{curl} \, F) = \nabla^2 F. \]

**Proof.** (a)

(b)

(c) [Problem Sheet 5, Question 1(vi)].

\[ \square \]

The operator

\[ \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  
(2.15)

in (2.11) and (2.14) is called the **Laplace–operator** (also denoted \( \Delta \)). It is very important in mathematical physics. Many of the basic partial differential equations (PDEs) of mathematical physics involve it, e.g. the **Laplace Equation**

\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \]  
(2.16)

We will come back to this equation in the second part of this course.
2.2.2 Application: Potential Theory [Bourne, pp. 225–243]

Definition 2.12. A vector field \( F : \Omega \to \mathbb{R}^3 \) is called \textit{irrotational}, if

\[
\text{curl} F = 0.
\]

In view of Equation (2.12) we can state the following

Proposition 2.13. Let \( F : \Omega \to \mathbb{R}^3 \) be a vector field. The following three statements are equivalent:

(a) \( F \) is irrotational.

(b) There exists a scalar field \( \phi : \Omega \to \mathbb{R} \) such that \( F = \nabla \phi \). \( \phi \) is called the \textit{scalar potential} of \( F \).

(c) \( F \) is conservative (i.e. work integral independent of path).

Proof. [Problem Sheets 2,4 & 8].

Remark 2.14. (Calculating scalar potentials). Let \( F : \Omega \to \mathbb{R}^3 \) be a conservative vector field and let \( \mathcal{C} \) be an arbitrary curve from \( \mathbf{a} \) to \( \mathbf{x} \). Then

\[
\phi(x) = \int_{\mathcal{C}} F \cdot d\mathbf{r}
\]

is a scalar potential for \( F \), if \( \mathbf{a} \) is not a pole of \( \phi \). See [Problem Sheet 4, Question 4].

Definition 2.15. A vector field \( F : \Omega \to \mathbb{R}^3 \) is called \textit{solenoidal} (or \textit{divergence-free}) if

\[
\text{div} F = 0.
\]

Example 2.16. (Application in Electrostatics). Given an electric field \( E \) and no sources, the \textbf{Maxwell’s Equations} state:

Remark 2.17. Let \( \Omega \) be bounded and simply connected [Bourne, pp. 225-226]. As a consequence of Equation (2.13) we have also (without proof):

(a) A vector field \( F : \Omega \to \mathbb{R}^3 \) is solenoidal iff there exists a vector field \( \Psi \) such that \( F = \text{curl} \Psi \). \( \Psi \) is called a \textit{vector potential} of \( F \) [Bourne, pp. 230–232].

(b) For every vector field \( F : \Omega \to \mathbb{R}^3 \) there exist a scalar field \( \phi \) and a vector field \( \Psi \) such that

\[
F = \text{grad} \phi + \text{curl} \Psi,
\]

i.e. any vector field can be resolved into the sum of an irrotational and a solenoidal part. This is the famous \textbf{Helmholtz Theorem} [Bourne, pp. 238–239].
2.3 Differentiation in Curvilinear Coordinates

[ Bourne, pp. 118–136 ]

Motivation. So far, in this chapter, we have only looked at scalar fields and vector fields in Cartesian coordinates \((x, y, z)\). What if we have curvilinear coordinates \((u, v, w)\) defined by

\[
\mathbf{r}(u, v, w) = x(u, v, w) \mathbf{i} + y(u, v, w) \mathbf{j} + z(u, v, w) \mathbf{k},
\]

e.g. spherical polar coordinates, etc.?

Note. In this section we will always use the notation \(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}\) for vectors in Cartesian coordinates rather than \((x, y, z)^T\). I will mention later on why.

We have already seen in Chapter 1 that varying \(u\) with \(v\) and \(w\) fixed creates a curve \(\mathcal{C}_u\) in \(\mathbb{R}^3\) with tangent vector \(\mathbf{r}_u := \partial \mathbf{r} / \partial u\). Similarly, varying \(v\) creates a curve \(\mathcal{C}_v\) with tangent vector \(\mathbf{r}_v := \partial \mathbf{r} / \partial v\), and varying \(w\) creates a curve \(\mathcal{C}_w\) with tangent vector \(\mathbf{r}_w := \partial \mathbf{r} / \partial w\):

\[2.3.1\] Orthogonal Curvilinear Coordinates

Definition 2.18. A triple \((u, v, w) \in D\) together with a Cartesian map \(\mathbf{r} : D \rightarrow \Omega \subset \mathbb{R}^3\) as defined in (2.19) is called a set of orthogonal curvilinear coordinates (OCCs) on \(\Omega\), if

\(a\) \(\mathbf{r}\) is a continuously differentiable bijection with continuously differentiable inverse \(\mathbf{r}^{-1}\) almost everywhere.

\[25\]
(b) The vectors $\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_w$ are mutually orthogonal, i.e.

\[ \mathbf{r}_u \cdot \mathbf{r}_v = \mathbf{r}_u \cdot \mathbf{r}_w = \mathbf{r}_v \cdot \mathbf{r}_w = 0. \]

Remark 2.19. Definition 2.18 is quite restrictive. In fact, a hard theorem in Topological Group Theory shows that there are only 11 OCC systems (ignoring translations, reflections, rotations and stretches).

Example 2.20. Spherical polar coordinates $(\rho, \theta, \phi)$:

\[
\begin{align*}
  x(\rho, \theta, \phi) &= \rho \sin \theta \cos \phi, \\
  y(\rho, \theta, \phi) &= \rho \sin \theta \sin \phi, \\
  z(\rho, \theta, \phi) &= \rho \cos \theta.
\end{align*}
\]

(a)

\[
\begin{align*}
  \text{Note.} \\
  \text{arg}(x, y) &:= \begin{cases} \\
  \tan^{-1}(y/x) & \text{for } x > 0, \ y \geq 0 \\
  \pi/2 & \text{for } x = 0, \ y > 0 \\
  \tan^{-1}(y/x) + \pi & \text{for } x < 0 \\
  3\pi/2 & \text{for } x = 0, \ y < 0 \\
  \tan^{-1}(y/x) + 2\pi & \text{for } x > 0, \ y < 0 \\
\end{cases}
\end{align*}
\]

(b)

\[
\begin{align*}
  \mathbf{r}_\rho &= \sin \theta \cos \phi \ \mathbf{i} + \sin \theta \sin \phi \ \mathbf{j} + \cos \theta \ \mathbf{k} \\
  \mathbf{r}_\theta &= \rho \cos \theta \cos \phi \ \mathbf{i} + \rho \cos \theta \sin \phi \ \mathbf{j} - \rho \sin \theta \ \mathbf{k} \\
  \mathbf{r}_\phi &= -\rho \sin \theta \sin \phi \ \mathbf{i} + \rho \sin \theta \cos \phi \ \mathbf{j}
\end{align*}
\]

DIY

\[
\begin{align*}
  \mathbf{r}_\rho \cdot \mathbf{r}_\theta &= \\
  \mathbf{r}_\rho \cdot \mathbf{r}_\phi &= \\
  \mathbf{r}_\theta \cdot \mathbf{r}_\phi &= 
\end{align*}
\]
It is useful to introduce unit vectors
\[ e_u := \frac{r_u}{|r_u|}, \quad e_v := \frac{r_v}{|r_v|}, \quad e_w := \frac{r_w}{|r_w|}. \] (2.20)

Let \( \mathbf{F} : \Omega \to \mathbb{R}^3 \) be a vector field on \( \Omega \). Since \( e_u, e_v, e_w \) are orthonormal, we can use them as a basis for representing \( \mathbf{F} \), i.e.
\[ \mathbf{F} = F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w \] (2.21)
where \( F_u, F_v, F_w \) are the components of \( \mathbf{F} \) along the coordinate lines \( \mathcal{C}_u, \mathcal{C}_v \) and \( \mathcal{C}_w \).

**Addition to previous note.** This is why we do not use the notation \( \mathbf{F} = (F_1, F_2, F_3)^T \) here. It does not carry any information on the coordinate system we work in.

How do we find \( F_u, F_v \) and \( F_w \)? Note that since \( e_u, e_v, e_w \) are orthonormal,
\[ \mathbf{F} \cdot \mathbf{e}_u = F_u \mathbf{e}_u \cdot \mathbf{e}_u + F_v \mathbf{e}_v \cdot \mathbf{e}_u + F_w \mathbf{e}_w \cdot \mathbf{e}_u = F_u. \] (2.22)

Similarly, \( \mathbf{F} \cdot \mathbf{e}_v = F_v \) and \( \mathbf{F} \cdot \mathbf{e}_w = F_w \).

**Example 2.21.** Express the vector field \( \mathbf{F} = z \mathbf{i} \) in spherical polar coordinate form.

\[
\begin{align*}
|r_\rho| &= \sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta} = 1 \\
|r_\theta| &= \sqrt{\rho^2 \{\cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + \sin^2 \theta\}} = \rho \\
|r_\phi| &= \sqrt{\rho^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi)} = \rho \sin \theta
\end{align*}
\]

\[
\begin{align*}
e_\rho &= \\
e_\theta &= \\
e_\phi &=
\end{align*}
\]

Thus
\[
\begin{align*}
F_\rho &= \mathbf{F} \cdot \mathbf{e}_\rho = \\
F_\theta &= \mathbf{F} \cdot \mathbf{e}_\theta = \\
F_\phi &= \mathbf{F} \cdot \mathbf{e}_\phi =
\end{align*}
\]

and
\[
\mathbf{F} =
\]
2.3.2 Grad, div, curl, \( \nabla^2 \) in Orthogonal Curvilinear Coordinates

Proposition 2.22. (a) \( \nabla f = \frac{1}{|r_u|} \frac{\partial f}{\partial u} e_u + \frac{1}{|r_v|} \frac{\partial f}{\partial v} e_v + \frac{1}{|r_w|} \frac{\partial f}{\partial w} e_w \) (2.23)

(b) \( \text{div} \, F = \frac{1}{|r_u||r_v||r_w|} \left\{ \frac{\partial}{\partial u} \left( |r_v||r_w| F_u \right) + \frac{\partial}{\partial v} \left( |r_u||r_w| F_v \right) + \frac{\partial}{\partial w} \left( |r_u||r_v| F_w \right) \right\} \) (2.24)

(c) \( \text{curl} \, F = \frac{1}{|r_u||r_v||r_w|} \left| \begin{array}{ccc} |r_u| e_u & |r_v| e_v & |r_w| e_w \\ \text{\( \partial / \partial u \)} & \text{\( \partial / \partial v \)} & \text{\( \partial / \partial w \)} \\ r_u F_u & r_v F_v & r_w F_w \end{array} \right| \) (2.25)

(d) \( \nabla^2 f = \frac{1}{|r_u||r_v||r_w|} \left\{ \frac{\partial}{\partial u} \left( \frac{|r_v||r_w|}{|r_u|} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{|r_u||r_w|}{|r_v|} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left( \frac{|r_u||r_v|}{|r_w|} \frac{\partial f}{\partial w} \right) \right\} \) (2.26)

Proof. (a)

(b) [Problem Sheet 6, Question 1].

(c) See [Bourne, pp. 131–132].

(d)
Example 2.23. For a scalar field $f : \Omega \to \mathbb{R}$ find $\nabla f$ and $\nabla^2 f$ in spherical polar coordinates.

Recall Example 2.21: $|r_\rho| = 1,$ $|r_\theta| = \rho,$ $|r_\phi| = \rho \sin \theta.$

Therefore

\begin{center}
\textbf{DIY Exercise.} Use spherical polar coordinates to show that $f(x) = |x|^{-1}$ satisfies the Laplace Equation $\nabla^2 f = 0.$
\end{center}

Example 2.24. Let $F := \rho^2 \cos \theta \ e_\rho \ + \ \frac{1}{\rho} \ e_\theta \ + \ \frac{1}{\rho \sin \theta} \ e_\phi$ in spherical polar coordinates. Find $\text{curl} F.$

\[
\text{curl} F = \frac{1}{\rho^2 \sin \theta} \begin{vmatrix}
  e_\rho & e_\theta & \rho \sin \theta \ e_\phi \\
  \partial/\partial \rho & \partial/\partial \theta & \partial/\partial \phi \\
  F_\rho & F_\theta & \rho \sin \theta F_\phi
\end{vmatrix}
\]

\[
= \frac{1}{\rho^2 \sin \theta} \left\{ \left( \frac{\partial(1)}{\partial \theta} - \frac{\partial(1)}{\partial \phi} \right) e_\rho + \left( \frac{\partial(1)}{\partial \rho} - \frac{\partial(\rho \sin \theta)}{\partial \phi} \right) \rho e_\theta + \left( \frac{\partial(1)}{\partial \rho} - \frac{\partial(\rho^2 \cos \theta)}{\partial \theta} \right) \rho \sin \theta e_\phi \right\}
\]

\[
= \rho \sin \theta \ e_\phi
\]