# MA20010: Vector Calculus and Partial Differential Equations

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## This course deals with

basic concepts and results in **vector integration** and **vector calculus**, **Fourier series**, and the solution of **partial differential equations** by **separation of variables**.

## It is

**fundamental** to most areas of applied Maths and many areas of pure Maths,

**a prerequisite** for a large number (**11!!**) of courses in Semester 2 and in Year 3/4.

## There will be two class tests:

**Tuesday, 31st October, 5.15pm** (Line, surface, volume integrals)

**Tuesday, 28th November, 5.15pm** (Vector calculus and integral theorems)

Questions will be similar to the questions on the problem sheets!

> **Please take this course seriously and do not fall behind with the problem sheets**. The syllabus for this course is **very dense** and only through **constant practice** will you be able to grasp the multitude of methods and concepts.

## Please revise

**MA10005 Sect. 6-8** Multivariate Calculus

**MA10006 Sect. 1** Vector Algebra

The first **AIM quiz** is designed to help you focus on the relevant material in those courses.

# Contents

# Chapter 1

# Vector Integration

In this course we will be dealing with functions in **more than one** unknown whose function value *might* be vector–valued as well.

**Definition 1.1.** A function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called a **vector field** on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a **scalar field** on $\mathbb{R}^n$. (Usually $n$ will be 2 or 3.)

**Example 1.2.** Typical examples of vector fields are

- gravitational field $\boldsymbol{G}(\boldsymbol{x})$
- electrical field $\boldsymbol{E}(\boldsymbol{x})$
- magnetic field $\boldsymbol{B}(\boldsymbol{x})$
- velocity field $\boldsymbol{V}(\boldsymbol{x})$

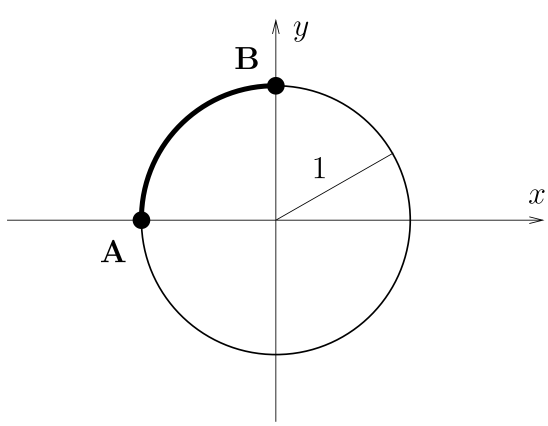
**Example 1.3.** Typical examples of scalar fields are

- speed $|\boldsymbol{V}(\boldsymbol{x})|$
- kinetic energy $\frac{1}{2}\, m\, |\boldsymbol{V}(\boldsymbol{x})|^2$
- electrical potential $\varphi_E(\boldsymbol{x})$

We will now learn how to integrate such fields in $\mathbb{R}^n$.

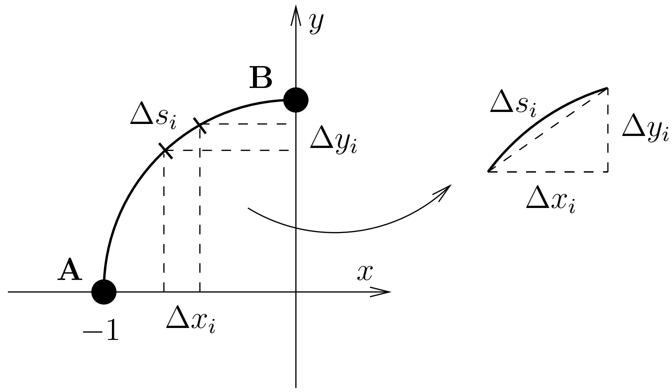
## 1.1 Line Integrals **[Bourne, pp 147–156] & [Anton, pp. 1100–1123]**

**Example 1.4. (Motivating Problem).** What is the perimeter of the unit circle?




Perimeter $= 4 * \text{length } s \text{ of arc } \mathbf{AB}$

Let us divide the interval $[-1, 0]$ into $N$ elements of length $\Delta x_i = 1/N$, $i = 0, \ldots, N-1$:




Obviously, with $\Delta s_i$ as constructed in the above figure

$$s \;=\; \sum_{i=0}^{N-1} \Delta s_i \;.$$

Moreover

$$s = \int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx \tag{1.1}$$

**DIY** Use the substitution $x = -\cos\theta$ to evaluate (1.1):

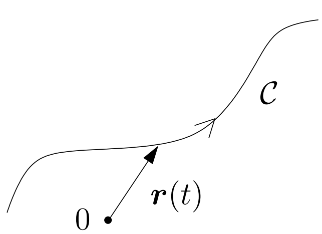
Hence, the perimeter is . . . . . . . . . . . . . . .

### 1.1.1 Parametric Representation – Arclength

Let $\mathcal{C}$ be a curve in $\mathbb{R}^n$ with **parametric representation** $\boldsymbol{r}(t)$, i.e.

$$\mathcal{C} = \{\boldsymbol{r}(t) \ : \ t_0 \le t \le t_e\} \tag{1.2}$$

where $\boldsymbol{r} : [t_0, t_e] \to \mathbb{R}^n$ is continuously differentiable.

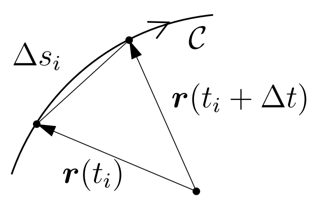



As $t$ increases $\boldsymbol{r}(t)$ traces out the curve $\mathcal{C}$.

**Note.** In Ex. 1.4 $t$ was chosen to be $x$. In general, $t$ can be any parameter (e.g. **angle** $\theta$).

**Example 1.5.** Give a parametric representation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

As in Example 1.4 we can now calculate the **arclength** $s(t)$ of $\mathcal{C}$ from $\boldsymbol{r}(t_0)$ to $\boldsymbol{r}(t)$ for every $t \in [t_0, t_e]$. Let $\Delta t := \dfrac{t - t_0}{N}$ for some $N \in \mathbb{N}$ and let $t_i := t_0 + i\Delta t$, for $i = 0, \ldots, N-1$.




Therefore

(1.3)

and

(1.4)

Letting $N \to \infty$ (which also means that $\Delta t \to 0$), the arclength is

$$s(t) \;=\; \int_{t_0}^{t} \left|\frac{d\boldsymbol{r}}{dt}\right| dt \qquad \left( = \int_{t_0}^{t} \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\, dt \quad \text{in } \mathbb{R}^3 \right). \tag{1.5}$$

Also, in the limit as $\Delta t \to 0$ equation (1.3) becomes

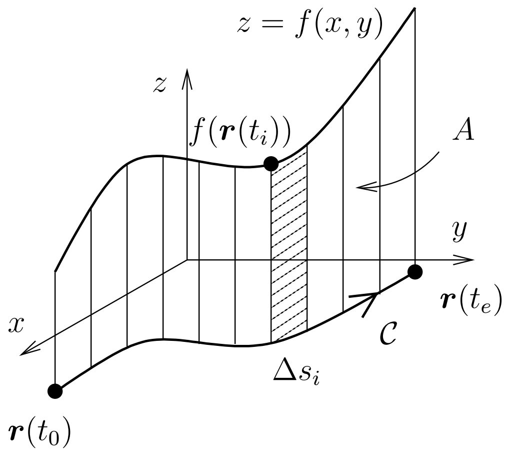
$$ds \;=\; \left|\frac{d\boldsymbol{r}}{dt}\right| dt\ . \tag{1.6}$$

The length of the entire curve $\mathcal{C}$ is given by

$$L_{\mathcal{C}} \;=\; s(t_e) \;=\; \int_{t_0}^{t_e} \left|\frac{d\boldsymbol{r}}{dt}\right| dt\ . \tag{1.7}$$

### 1.1.2 Line Integrals of Scalar Fields

Now assume we want to calculate the integral of a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ along the curve $\mathcal{C}$. Geometrically this means, to calculate the following area $A$:




(in $\mathbb{R}^2$)

As in (1.4) with $\Delta t = \dfrac{t_e - t_0}{N}$ we get

(i.e. we approximate the area by the sum of the strips.)

Hence for $N \to \infty$

$$A = \int_{t_0}^{t_e} f(\boldsymbol{r}(t)) \left| \frac{d\boldsymbol{r}}{dt} \right| dt$$

(an ordinary integral of a scalar valued function of t).

**Definition 1.6.** The **line integral of a scalar field** $f : \mathbb{R}^n \to \mathbb{R}$ along the curve $\mathcal{C}$ in (1.2) is defined as

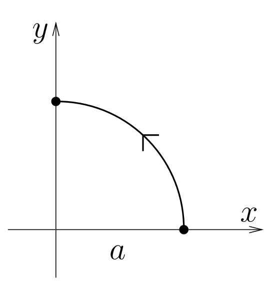
$$\int_{\mathcal{C}} f \, ds \; := \; \int_{t_0}^{t_e} f(\boldsymbol{r}(t)) \left| \frac{d\boldsymbol{r}}{dt} \right| dt \; . \tag{1.8}$$

*Remark* 1.7.

(a) The value of $\int_{\mathcal{C}} f \, ds$ does not depend on the choice of parametrisation, even if the orientation of $\mathcal{C}$ is reversed [Bourne, p. 148], [Anton, pp. 1108–1109].

(b) The length of the curve $\mathcal{C}$ in (1.6) is a special line integral, i.e. $L_{\mathcal{C}} = \int_{\mathcal{C}} 1 \, ds$.

**Example 1.8.** Integrate $f(x, y) = 2xy$ around the first quadrant of a circle with radius $a$ as shown:



**Step 1.** Parametrise circle:

**Step 2.** Calculate the line element $ds$ :

**DIY** **Step 3.** Apply formula (1.8) for the line integral:

### 1.1.3 Line Integrals of Vector Fields

**Definition 1.9.** The **work integral of a vector field** $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ along the curve $\mathcal{C}$ in (1.2) is defined as

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} \;:=\; \int_{t_0}^{t_e} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}\, dt \;. \tag{1.9}$$

(**dot product!**)

**Theorem 1.10.** *If $\hat{\boldsymbol{T}}$ is the unit tangent vector to $\mathcal{C}$ in (1.2) that points in the direction in which $t$ is increasing, then*

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} \;=\; \int_{\mathcal{C}} (\boldsymbol{F} \cdot \hat{\boldsymbol{T}})\, ds \;, \tag{1.10}$$

*i.e. the work integral of $\boldsymbol{F}$ = the line integral of the component of $\boldsymbol{F}$ parallel to $\mathcal{C}$.*

*Proof.*

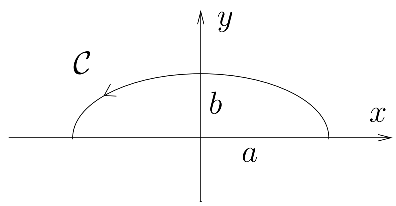
□

*Remark* 1.11.

(a) It follows directly from Theorem 1.10 that a reversal of the orientation of $\mathcal{C}$ **does** change the sign of the work integral (in contrast to the line integral of a scalar field, cf. Remark 1.7(a)) [Bourne, p. 153], [Anton, p. 1109].

(b) If $\boldsymbol{F}$ is a force field then $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ is the **work** done by moving a particle from $\boldsymbol{r}(t_0)$ to $\boldsymbol{r}(t_e)$ along the curve $\mathcal{C}$; hence the term *work integral*.

**Example 1.12. (in tutorial).** Evaluate the integral $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x, y) = (2y, x^2)^T$ and $\mathcal{C}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in the upper half plane as shown:




**Step 1.** Parametrise the curve (Example 1.5):

$$\boldsymbol{r}(t) = (a\cos t, b\sin t)^T, \quad t \in [0, \pi].$$

**Step 2.** Calculate the **vector line element** $d\boldsymbol{r} = \dfrac{d\boldsymbol{r}}{dt}\, dt$ :

**Step 3.** Apply formula (1.9) for the work integral:

**Definition 1.13. (in tutorial).** Let $\boldsymbol{F}$ be a vector field such that $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$. The (vector valued) integrals $\int_\mathcal{C} \boldsymbol{F}\, ds$ and $\int_\mathcal{C} \boldsymbol{F} \wedge d\boldsymbol{r}$ are defined as

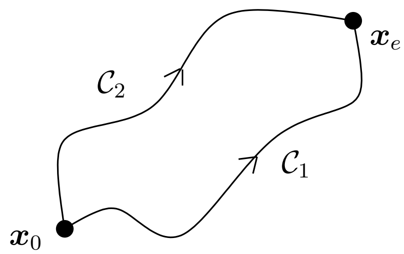
$$\int_\mathcal{C} \boldsymbol{F}\, ds \;:=\; \left(\int_\mathcal{C} F_1\, ds, \int_\mathcal{C} F_2\, ds, \int_\mathcal{C} F_3\, ds\right)^T$$

$$\int_\mathcal{C} \boldsymbol{F} \wedge d\boldsymbol{r} \;:=\; \int_{t_0}^{t_e} \boldsymbol{F}(\boldsymbol{r}(t)) \wedge \frac{d\boldsymbol{r}}{dt}\, dt\ .$$

**Definition 1.14.** A vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called **conservative**, if

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{r} \;=\; \int_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{r} \tag{1.11}$$

for any two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ connecting two points $\boldsymbol{x}_0$ and $\boldsymbol{x}_e$ in $\mathbb{R}^n$




### 1.1.4 Application to Particle Motion

Suppose $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a force field and suppose that a particle of mass $m$ moves along a curve $\mathcal{C}$ through this field. Suppose further that $\boldsymbol{r}(t)$ is the position of the particle at time $t \in [t_0, t_e]$ (i.e. $\boldsymbol{r}(t)$ is a parametric representation of $\mathcal{C}$ where the parameter $t$ is time).

Recall:

- particle velocity $\quad \boldsymbol{V}(t) \;=\; \dfrac{d\boldsymbol{r}}{dt}$
- kinetic energy $\quad K(t) \;=\; \frac{1}{2}m\,|\boldsymbol{V}(t)|^2$

Newton's Second Law states

$$\boxed{\text{Force applied} \;=\; \text{mass} \;*\; \text{acceleration}} \qquad \Longrightarrow \qquad \boldsymbol{F}(\boldsymbol{r}(t)) \;=\; m\,\frac{d^2\boldsymbol{r}}{dt^2}\ .$$

Therefore

$$\begin{aligned}\int_\mathcal{C} \boldsymbol{F} \cdot d\boldsymbol{r} \;&=\; \int_{t_0}^{t_e} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}\, dt \;=\; \int_{t_0}^{t_e} \left(m\frac{d^2\boldsymbol{r}}{dt^2}\right) \cdot \frac{d\boldsymbol{r}}{dt}\, dt \\ &=\; \frac{1}{2}m \int_{t_0}^{t_e} \frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt} \cdot \frac{d\boldsymbol{r}}{dt}\right) dt \;=\; \frac{1}{2}m \int_{t_0}^{t_e} \frac{d}{dt}|\boldsymbol{V}(t)|^2\, dt\end{aligned}$$

and so by the Fundamental Theorem of Calculus

$$\int_\mathcal{C} \boldsymbol{F} \cdot d\boldsymbol{r} \;=\; \frac{1}{2}m\,|\boldsymbol{V}(t_e)|^2 \;-\; \frac{1}{2}m\,|\boldsymbol{V}(t_0)|^2\ , \tag{1.12}$$

or in other words

| Work done = Change in kinetic energy from $t_0$ to $t_e$ |
|---|

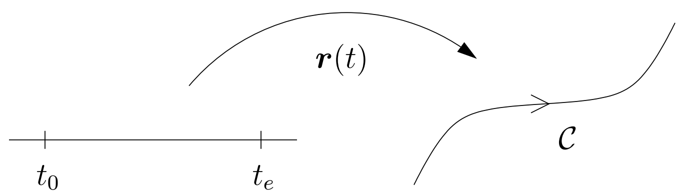
(**"conservation of energy principle"**).

## 1.2 Surface Integrals [Bourne, pp. 172–189] & [Anton, pp. 1130–1145]

In the previous section we discussed integrals along "one-dimensional" curves in $\mathbb{R}^n$. Now we want to look at integrals on **two-dimensional surfaces** in $\mathbb{R}^n$.

You have already done the case $n = 2$ in MA10005, i.e. (flat) surfaces in $\mathbb{R}^2$. Now we will consider surfaces in $\mathbb{R}^3$, i.e. $n = 3$.

Recall (for curves): parametric representation

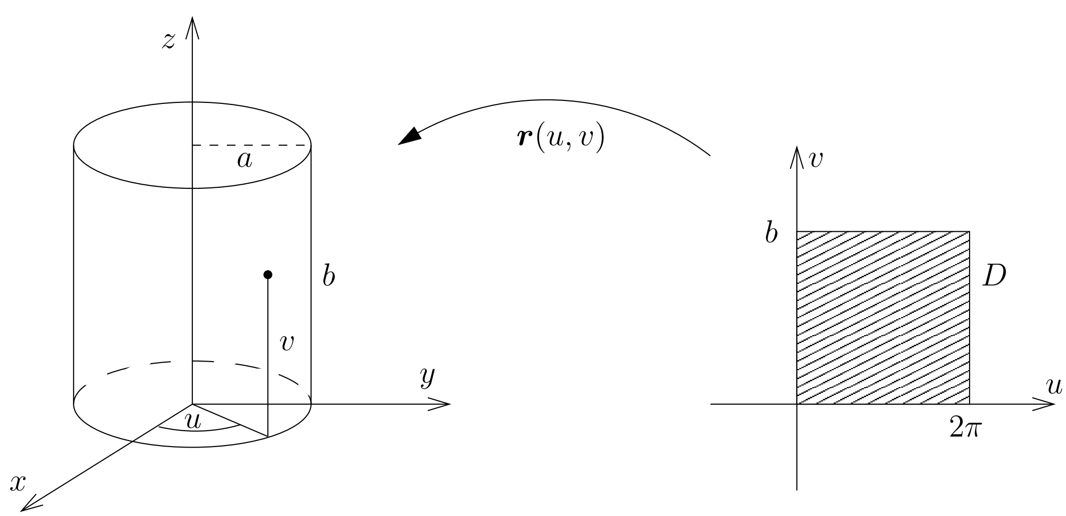



Now, let $\mathcal{S}$ be a (two-dimensional) surface in $\mathbb{R}^3$ with **parametric representation** $\boldsymbol{r}(u,v)$, i.e.

$$\mathcal{S} \;=\; \{\boldsymbol{r}(u,v) : (u,v) \in D\} \tag{1.13}$$

where $\boldsymbol{r} : D \to \mathbb{R}^3$ is continuously differentiable and $D \subset \mathbb{R}^2$.

**Example 1.15. (Step 1).** A cylindrical shell of height $b$ and radius $a$




can be parametrised by

### 1.2.1 The Surface Elements $dS$ and $\boldsymbol{dS}$

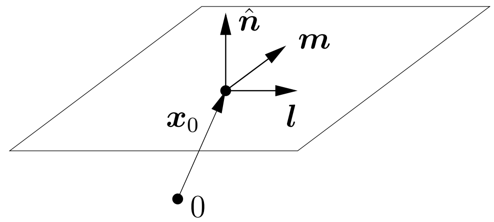
**Recall:** the equation of a plane in $\mathbb{R}^3$ is

$$\boldsymbol{x} = \boldsymbol{x}_0 + \lambda \boldsymbol{l} + \mu \boldsymbol{m} , \qquad \lambda, \mu \in \mathbb{R}$$

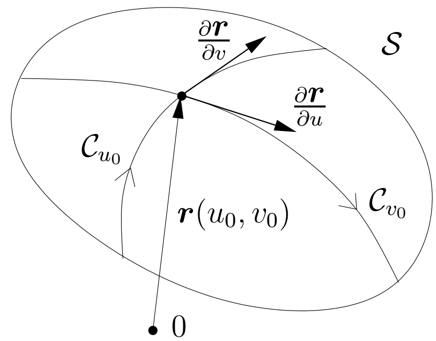
(where $\boldsymbol{l}$, $\boldsymbol{m}$ are two arbitrary non-parallel vectors on the plane).

The unit normal to this plane is

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{l} \wedge \boldsymbol{m}}{|\boldsymbol{l} \wedge \boldsymbol{m}|} .$$




Now, let $\mathcal{S} = \{\boldsymbol{r}(u,v) : (u,v) \in D\}$ as in (1.13) and let $(u_0, v_0) \in D$. Fixing $v_0$ we get




Similarly,

**Definition 1.16.** (a) The tangent plane of $\mathcal{S}$ in (1.13) at the point $\boldsymbol{r}(u_0, v_0) \in \mathcal{S}$ is

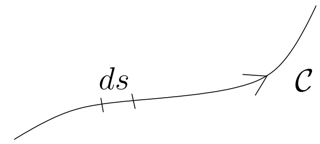
$$T := \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} = \boldsymbol{r}(u_0, v_0) + \lambda \frac{\partial \boldsymbol{r}}{\partial u}(u_0, v_0) + \mu \frac{\partial \boldsymbol{r}}{\partial v}(u_0, v_0) , \ \lambda, \mu \in \mathbb{R} \right\} . \qquad (1.14)$$

(b) The unit normal to $\mathcal{S}$ at $\boldsymbol{r}(u_0, v_0)$ is

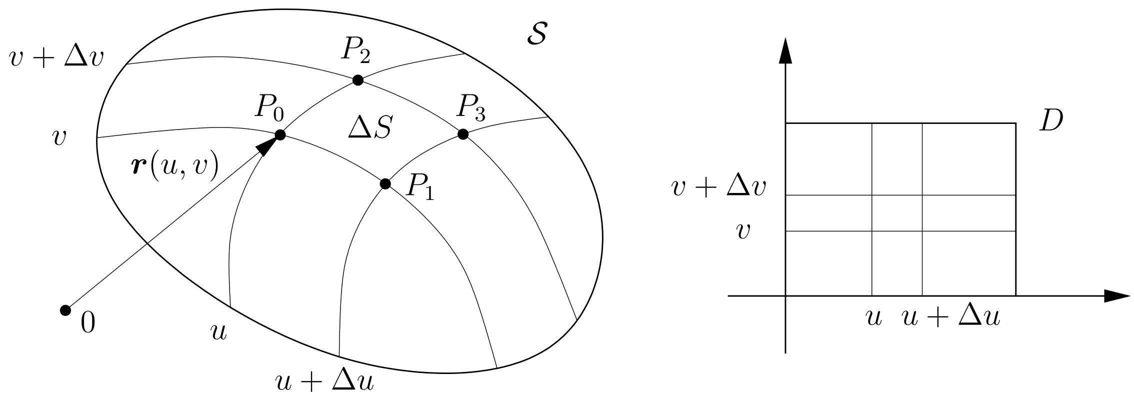
$$\hat{\boldsymbol{n}} := \left( \frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v} \right) \Big/ \left| \frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v} \right| (u_0, v_0) \qquad (1.15)$$

**Note (on orientation).** Reversing the rôles of $u$ and $v$, changes the sign of $\hat{\boldsymbol{n}}$. This is a matter of **convention**. For closed surfaces (e.g. sphere, cylinder, torus) it is common to define $\hat{\boldsymbol{n}}$ to be the normal that points **outward**. (If it is not clear we will always specify the orientation of the normal/surface.)

We will now define a surface element $dS$ for the surface $\mathcal{S} = \{\boldsymbol{r}(u,v) : (u,v) \in D\}$ in (1.13): Recall the line element $ds := \left|\frac{d\boldsymbol{r}}{dt}\right| dt$ which we defined in (1.6) for a (one-dimensional) curve $\mathcal{C}$.




Let $P_0$ be a point on the surface, such that $\vec{OP_0} = \boldsymbol{r}(u,v)$, let $P_1$ be a neighbouring point with $\vec{OP_1} = \boldsymbol{r}(u + \Delta u, v)$. Similarly, let $P_2$ and $P_3$ be the points with $\vec{OP_2} = \boldsymbol{r}(u, v + \Delta v)$ and $\vec{OP_3} = \boldsymbol{r}(u + \Delta u, v + \Delta v)$.




For $\Delta u$ and $\Delta v$ sufficiently small

$$dS \;=\; \left|\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right| du\, dv\ . \tag{1.16}$$

$dS$ is called the **(scalar) surface element**.

**Example 1.17. (Step 2).** Calculate $dS$ and find the outward unit normal of the cylindrical shell of height $b$ and radius $a$.

**DIY** (Recall **Step 1** (i.e. Example 1.15): $\boldsymbol{r}(u,v) := (a\cos u, a\sin u, v)^T$.)

The **vector surface element** $\boldsymbol{dS}$ is defined as

$$\boldsymbol{dS} := dS\,\hat{\boldsymbol{n}} , \tag{1.17}$$

or using (1.15) and (1.16)

$$\boldsymbol{dS} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) du\,dv . \tag{1.18}$$

### 1.2.2 Surface Integrals of Scalar Fields

**Definition 1.18.** The **surface integral of a scalar field** $f : \mathbb{R}^n \to \mathbb{R}$ on $\mathcal{S}$ in (1.13) is defined as

$$\iint_{\mathcal{S}} f\,dS = \iint_D f(\boldsymbol{r}(u,v)) \left|\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right| du\,dv . \tag{1.19}$$

**Example 1.19. (Step 3).** Find the surface area of a cylindrical shell of height $b$, radius $a$.

[Recall: **Step 1** (i.e. Example 1.15: $\boldsymbol{r}(u,v) := (a\cos u, a\sin u, v)^T$, $0 \le u \le 2\pi$, $0 \le v \le b$,

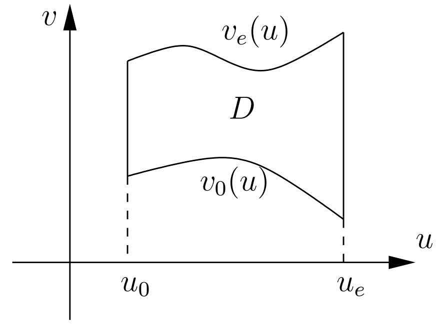
**Step 2** (i.e. Example 1.17: $dS = a\,du\,dv$ . ]

( *surface integral* $\longrightarrow$ *2D–integral* $\longrightarrow$ *repeated integral.* )

**Note.** Try always to parametrise a surface $\mathcal{S}$ in such a way that either $D$ is rectangular **or**

$$D = \{(u,v) : u \in [u_0, u_e] \text{ and } v \in [v_0(u), v_e(u)]\} .$$

This makes the transition from the 2D-integral to the repeated integral easier (compare MA10005).

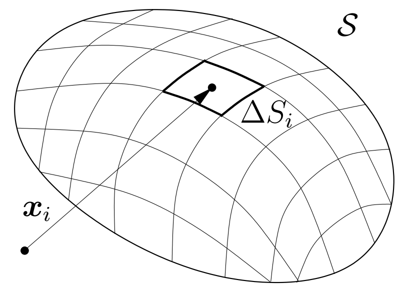



*Remark* 1.20.

(a) Another way to define the surface integral would be to use a subdivision of $\mathcal{S}$ into (little) surface elements $\Delta S_i$ with centre $\boldsymbol{x}_i$ (as for line integrals), i.e.

$$\iint_{\mathcal{S}} f\, dS \;=\; \lim_{N\to\infty} \sum_{i=0}^{N-1} f(\boldsymbol{x}_i)\, \Delta S_i \;. \qquad (1.20)$$

See [Anton, p. 1130].




(b) The area of a surface $\mathcal{S}$ is a special surface integral, i.e. $A_{\mathcal{S}} = \iint_{\mathcal{S}} dS$ (cf. Remark 1.7(b)).

### 1.2.3 Flux – Surface Integrals of Vector Fields

**Definition 1.21.** The **flux (or surface integral) of a vector field** $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ across a surface $\mathcal{S}$ is defined as

$$\iint_{\mathcal{S}} \boldsymbol{F}\cdot \boldsymbol{dS} \;=\; \iint_{D} \boldsymbol{F}(\boldsymbol{r}(u,v))\cdot\left(\frac{\partial \boldsymbol{r}}{\partial u}\wedge\frac{\partial \boldsymbol{r}}{\partial v}\right) du\, dv \;. \qquad (1.21)$$

*Remark* 1.22. (a) It follows from (1.17) that

$$\iint_{\mathcal{S}} \boldsymbol{F}\cdot \boldsymbol{dS} \;=\; \iint_{\mathcal{S}} (\boldsymbol{F}\cdot\hat{\boldsymbol{n}})\, dS \;,$$

i.e. the flux of $\boldsymbol{F}$ across $\mathcal{S}$ = surface integral of the normal component of $\boldsymbol{F}$ on $\mathcal{S}$.

(b) (physical interpretation) If $\boldsymbol{F}$ is for example the velocity field of some fluid then the flux across $\mathcal{S}$ is the net volume of fluid that passes through the surface per unit of time. For more details:

| **DIY** | Please study [Anton, pp. 1137–1140] until the next lecture. |
|---|---|

**Example 1.23.** Find $\iint_{\mathcal{S}} \boldsymbol{x}\cdot\boldsymbol{dS}$ , where $\mathcal{S}$ is the sphere with radius $a$, i.e.

$$S := \{\boldsymbol{x}\in\mathbb{R}^3 : x^2+y^2+z^2 = a^2\}.$$

**Step 1.** Parametrise $\mathcal{S}$: **spherical polar coordinates**

**Step 2.** Calculate $\dfrac{\partial \boldsymbol{r}}{\partial \theta} \wedge \dfrac{\partial \boldsymbol{r}}{\partial \varphi}$ :

**Step 3.** Apply formula (1.21) for the surface integral:

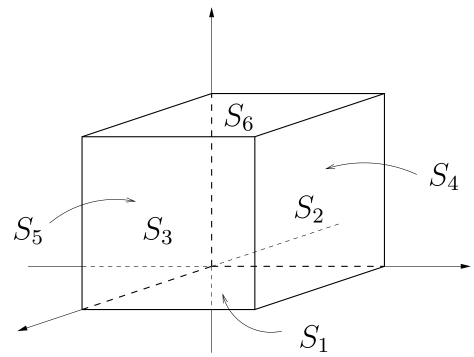
*Remark* 1.24. (a) So far we have only considered **smooth surfaces**, i.e. surfaces $\mathcal{S}$ with continuously differentiable parametric representation $\boldsymbol{r}$. However, the notion of surface integrals can be extended to surface integrals on **piecewise smooth surfaces**:

If $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_n$ , where $\mathcal{S}_i$ is smooth for each $i = 1, \ldots, n$, then we define

$$\iint_{\mathcal{S}} := \iint_{\mathcal{S}_1} + \iint_{\mathcal{S}_2} + \ldots + \iint_{\mathcal{S}_n} .$$

For example: Let $\mathcal{S}$ be the surface of the unit cube and let $\mathcal{S}_1, \ldots, \mathcal{S}_6$ be the six faces (obviously each one of the $\mathcal{S}_i$, $i = 1, \ldots, 6$ is smooth), and so

$$\iint_{\mathcal{S}} = \iint_{\mathcal{S}_1} + \iint_{\mathcal{S}_2} + \ldots + \iint_{\mathcal{S}_6} .$$




(b) Analagously, for **piecewise smooth curves** $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_m$ , we define

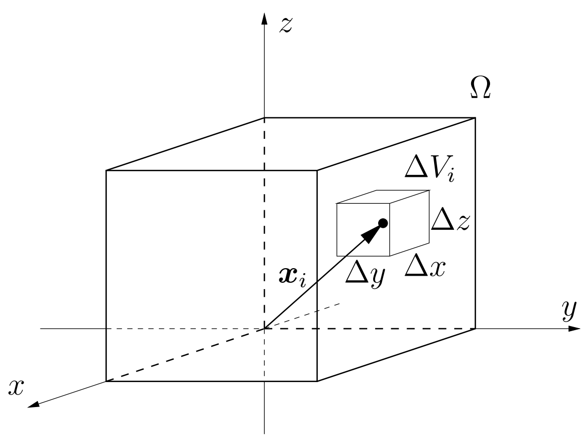
$$\int_{\mathcal{C}} := \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \ldots + \int_{\mathcal{C}_m} .$$

## 1.3 Volume Integrals **[Bourne, pp. 189–194] & [Anton, pp. 1048–1090]**

Let $f : \Omega \to \mathbb{R}$ be a scalar field defined on a **bounded region** $\Omega \subset \mathbb{R}^3$. As in $\mathbb{R}^2$ (recall MA10005) we can subdivide $\Omega$ into little volume elements $\Delta V_i$ with centre $\boldsymbol{x}_i$ and define the volume integral

$$\iiint_{\Omega} f\, dV = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(\boldsymbol{x}_i)\, \Delta V_i$$

(where $\Delta V_i \to 0$ as $N \to \infty$). In rectangular Cartesian coordinates one might choose $\Delta V_i = \Delta x\, \Delta y\, \Delta z$ and hence obtain in the limit the **volume element** $dV = dx\, dy\, dz$.



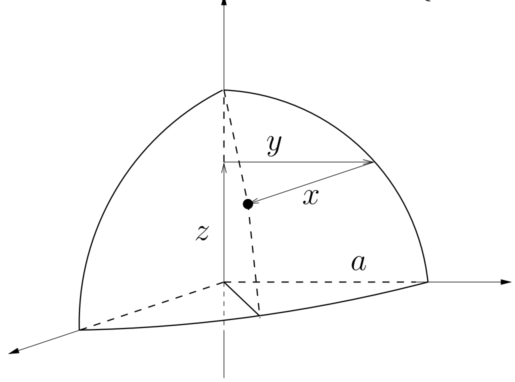

Thus

$$\iiint_{\Omega} f\, dV = \iiint_{\Omega} f(x, y, z)\, dx\, dy\, dz . \qquad (1.22)$$

If $f$ is continuous, the right hand side can be evaluated as a repeated integral again (in any order you want). However the limits of these repeated integrals **might be very akward**.

**Example 1.25.** Find the volume of the ball of radius $a$ in the first octant, i.e.

$$\Omega := \{\boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le a^2,\ x, y, z \ge 0\}.$$



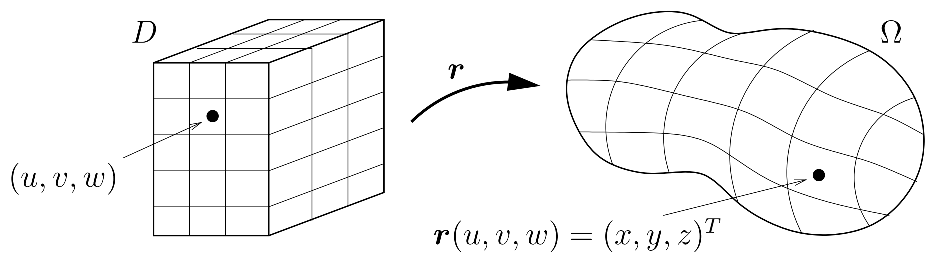
Things may become **even nastier** if we want to integrate over the entire ball.

### 1.3.1 Change of Variables – Reparametrisation

Let

$$\Omega \;:=\; \{\boldsymbol{r}(u,v,w) : (u,v,w) \in D\} \tag{1.23}$$

where $D \subset \mathbb{R}^3$ and $\boldsymbol{r} : D \to \Omega$ is continuously differentiable. This will be particularly uesful, if $D$ is box-shaped:




**Example 1.26.** The parametrisation

$$\boldsymbol{r}(\rho,\theta,\phi) \;:=\; (\rho\sin\theta\cos\phi\,,\ \rho\sin\theta\sin\phi\,,\ \rho\cos\theta)^T$$

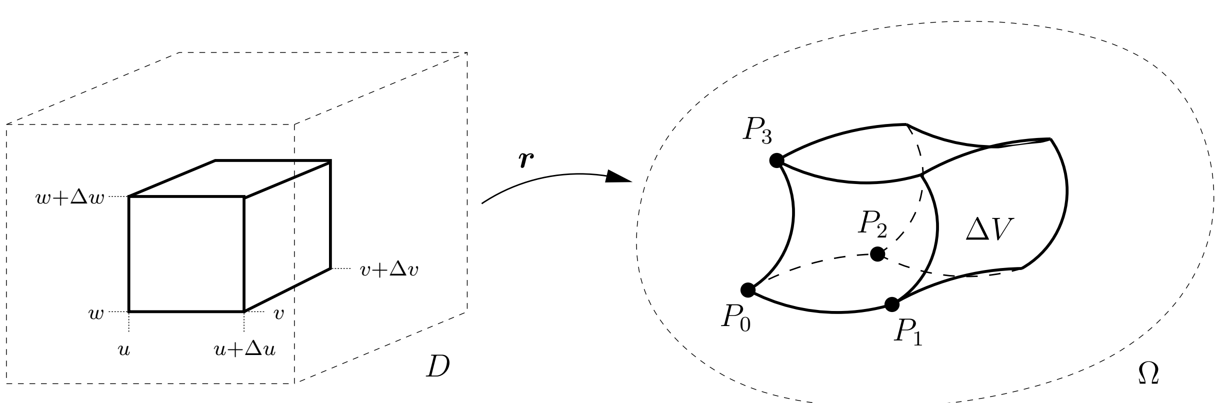
maps the (box-shaped) region

$$D \;=\; \{(\rho,\theta,\phi) \in \mathbb{R}^3 : 0 \le \rho \le a\,,\ 0 \le \theta \le \pi\,,\ 0 \le \phi < 2\pi\,\}$$

to the **ball** with radius $a$. $(\rho,\theta,\phi)$ are called **spherical polar coordinates** ($\rho\ldots$ radial distance, $\theta\ldots$ polar angle, $\phi\ldots$ azimuthal angle). **See handout!**

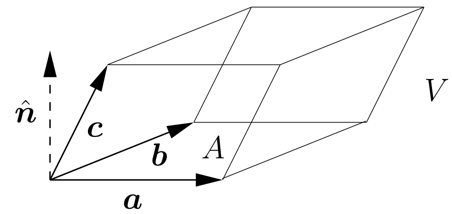
| **DIY** | Please study the handouts on spherical and cylindrical polar coordinates until the next lecture. |
|---|---|

**Motivation.** Let $\vec{OP_0} = \boldsymbol{r}(u,v,w)$, $\vec{OP_1} = \boldsymbol{r}(u+\Delta u,v,w)$, $\vec{OP_2} = \boldsymbol{r}(u,v+\Delta v,w)$, $\vec{OP_3} = \boldsymbol{r}(u,v,w+\Delta w)$.



For $\Delta u, \Delta v, \Delta w$ sufficiently small $\Delta V$ will be approximately a **parallelepiped**.

Recall (MA10006):




and in the limit as $\Delta u, \Delta v, \Delta w \to 0$

$$dV \;=\; \left| \left( \frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v} \right) \cdot \frac{\partial \boldsymbol{r}}{\partial w} \right| du\, dv\, dw \tag{1.24}$$

the so-called **volume element**.

**Definition 1.27.** The **volume integral of a scalar field** $f : \Omega \to \mathbb{R}$ over the region $\Omega \subset \mathbb{R}^3$ in (1.23) is defined as

$$\iiint_\Omega f\, dV \;=\; \iiint_D f(\boldsymbol{r}(u,v,w)) \left| \left( \frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v} \right) \cdot \frac{\partial \boldsymbol{r}}{\partial w} \right| du\, dv\, dw\ . \tag{1.25}$$

**Note.** Definition 1.27 is **no** contradiction to the **change of variables formula** which you have learnt in MA10005, since

$$\left|\left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) \cdot \frac{\partial \boldsymbol{r}}{\partial w}\right| = \det \begin{pmatrix} \frac{\partial r_1}{\partial u} & \frac{\partial r_2}{\partial u} & \frac{\partial r_3}{\partial u} \\ \frac{\partial r_1}{\partial v} & \frac{\partial r_2}{\partial v} & \frac{\partial r_3}{\partial v} \\ \frac{\partial r_1}{\partial w} & \frac{\partial r_2}{\partial w} & \frac{\partial r_3}{\partial w} \end{pmatrix} =: \frac{\partial(x,y,z)}{\partial(u,v,w)} \tag{1.26}$$

i.e. the **Jacobian determinant**.

[ **Proof.** (recall MA10006)

*Remark* 1.28.

(a) $u, v, w$ can be regarded as **curvilinear coordinates** on $\Omega$. (See the figure at the beginning of Section 1.3.1.)

(b) The integral (1.22) in Cartesian coordinates is just a special case with $\boldsymbol{r}(x,y,z) = (x,y,z)^T$, and so $\left|\left(\frac{\partial \boldsymbol{r}}{\partial x} \wedge \frac{\partial \boldsymbol{r}}{\partial y}\right) \cdot \frac{\partial \boldsymbol{r}}{\partial z}\right| = 1$.

(c) The volume of $\Omega$ in (1.23) is a special volume integral, i.e. $V_\Omega = \iiint_\Omega dV$ .

**Example 1.29.** Redo Example 1.25 using spherical polar coordinates.

**Step 1.** Parametrise (see handout):

**Step 2.** Calculate the volume element $dV = \left|\left(\frac{\partial \boldsymbol{r}}{\partial u} \wedge \frac{\partial \boldsymbol{r}}{\partial v}\right) \cdot \frac{\partial \boldsymbol{r}}{\partial w}\right| \, du\, dv\, dw$ :

**Step 3.** Evaluate Formula (1.25) for the volume integral: