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Abstract. This paper is on the construction of energy minimizing coarse spaces that obey certain functional constraints and can thus be used for example to build robust coarse spaces for elliptic problems with large variations in the coefficients. In practice they are built by patching together solutions to appropriate local saddle point or eigenvalue problems. We develop an abstract framework for such constructions, and then apply it in the design of coarse spaces for discretizations of PDEs with highly varying coefficients. The stability and approximation bounds of the constructed interpolant are in the weighted L_2 norm and are independent of the variations in the coefficients. Such spaces can be used for example in two level overlapping Schwarz algorithms for elliptic PDEs with large coefficient jumps generally not resolved by a standard coarse grid, or for numerical upscaling purposes. Some numerical illustration is provided.

1. Introduction

This paper is on the construction of energy minimizing coarse spaces that obey certain functional constraints and stable interpolation operators on those spaces. The specific application we have in mind are discretizations of scalar elliptic partial differential equations with large variations in the coefficients. The proposed technique can be used for example to design a uniformly convergent two–level Schwarz preconditioner.

Earlier works on the subject are usually under the assumption that the discontinuities are resolved by a coarsest grid. Under such condition, for the AMLI (Algebraic Multi Level Iteration) method proposed in [28], but in the traditional MG setting (for details, see [29, Section 5.6]) uniform condition number bounds can be shown for problems in two and three spatial dimensions with respect to both the coefficient variation and the mesh size. More recent works on nearly optimal estimates for multigrid preconditioners under the assumption of resolving the coefficient discontinuities are also found in [31, 34]. Theoretical results on the convergence of the overlapping Schwarz method in the case of resolved coefficients are found in the survey article [6] and in the monograph [26]. Under the additional assumption that the coefficient is *quasi-monotone*, bounds on the convergence rate of the overlapping Schwarz method are given in [11]. As shown in earlier works, if the quasi-monotonicity assumption on the coefficient is omitted, one needs to use carefully designed coarse spaces (a.k.a. "exotic" coarse spaces, see e.g. [11, 23]).

There are two important issues that we address in this paper. Firstly, an important feature of the methods that we propose is that there is no requirement to align a coarse

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grid with the discontinuities in the PDE coefficient. There has been some recent activity in the analysis of existing methods and in the design of new methods in this context (see e.g. [17, 24, 15, 10, 12, 13, 27, 14, 25], as well as [29] and the references therein, in the context of Algebraic Multigrid). Of course, there are restrictions in the sense that our algorithm may result in a coarse space of high dimension if the number of discontinuities in the coefficient is large. It should be noted however, that we do not require quasi-monotone coefficient distribution and in fact the resulting two level Schwarz method will be uniformly convergent under very general assumptions on the coefficient behavior. The key ingredient, which makes this possible is a construction of bases that preserve averages of the interpolated functions over prescribed regions.

Secondly, we have set up a framework, that allows us to handle in a unified fashion not only this type of functional constraints (preserving the averages), but also other types of functional constraints (see e.g., [30]). For example, coarse spaces that can be built up by preserving local eigenvectors, a technique recently proposed in [15, 13] are easily analyzed in a analogous fashion using our framework (cf. [5, 7] for earlier work in this direction). The abstract framework can also be applied to systems of PDEs (e.g. linear elasticity with large variations in material coefficients), but this would go beyond the scope of this paper.

The rest of the paper is organized as follows. We begin with some preliminaries in §2, mainly to set up the specific application we would like the reader to bear in mind, i.e. robust coarse spaces for two-level Schwarz methods for elliptic PDEs. In particular, we show which theoretical tools are needed to guarantee uniform convergence. We then prove, in §3, a fundamental approximation and stability result formulated in an abstract setting. In §4 we apply our abstract approximation result to 2nd–order elliptic problems with highly varying coefficients and design two types of coarse spaces with the desired local weak approximation and stability properties. In §5, we use a partition of unity to patch together the local coarse spaces and obtain a global coarse space with the same weak approximation and stability properties. Ways to reduce the dimension of the constructed coarse spaces by using new weighted Poincaré inequalities proved in [22, 21], which apply for locally quasi–monotone coefficients, are briefly discussed in §6. We conclude in §7 with some numerical results.

2. Preliminaries and notation

2.1. Model problem and discretization. We consider the variational formulation of a second order, elliptic boundary value problem with Dirichlet boundary conditions: Find $u^* \in H^1_0(\Omega)$, for a given polygonal (polyhedral) domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and a source term $f \in L_2(\Omega)$, such that

$$\underbrace{\int_\Omega \alpha(\boldsymbol{x})\ \nabla u^* \cdot \nabla v}_{\equiv a(u^*,\, v)} \;=\; \underbrace{\int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x})}_{\equiv (f,\, v)}\,, \quad \text{for all} \quad v \in H^1_0(\Omega). \tag{2.1}$$

We are interested in the case when the diffusion coefficient $\alpha = \alpha(\boldsymbol{x})$ is a piecewise constant function, that may have large variations within Ω. We thus assume that $\bar{\Omega} = \cup_{l=1}^m \bar{\mathcal{Y}}_l$, with polygonal (polyhedral) subdomains $\mathcal{Y}_l$, and that $\alpha(\boldsymbol{x}) = \alpha_l$, for all $\boldsymbol{x} \in \mathcal{Y}_l$ and $l = 1, \ldots, m$.

For any domain $D \subset \Omega$ we have the following energy norm

$$\|v\|^2_{a,D} = \int_D \alpha(\boldsymbol{x}) |\nabla v|^2 = \sum_{l=1}^m \alpha_l \int_{D\cap\mathcal{Y}_l} |\nabla v|^2. \tag{2.2}$$

We note that if $v \in H_0^1(D)$ this is indeed a norm, and for $v \in H^1(D)$ this is only a seminorm. We denote the seminorm by $|\cdot|_{a,D}$. We also need the weighted L_2 norm

$$\|v\|_{0,\alpha,D}^2 = \int_D \alpha(\boldsymbol{x}) v^2 = \sum_{l=1}^{m} \alpha_l \int_{D\cap\mathcal{Y}_l} v^2 \tag{2.3}$$

When $D = \Omega$ we omit the domain from the subscript and write $\|\cdot\|_a$ and $\|\cdot\|_{0,\alpha}$ instead of $\|\cdot\|_{a,\Omega}$ and $\|\cdot\|_{0,\alpha,\Omega}$, respectively. In addition, we also need the usual unweighted norms, which in standard notation we denote by

$$\|v\|_{L_2(D)}^2 = \int_D v^2, \quad |v|_{H^1(D)}^2 = \int_D |\nabla v|^2, \quad \|v\|_{H^1(D)}^2 = \|v\|_{L_2(D)}^2 + |v|_{H^1(D)}^2.$$

The corresponding inner products are denoted in the same way, e.g.

$$(v, w)_{L_2(D)} = \int_D vw \quad \text{or} \quad (v, w)_{0,\alpha,D} = \int_D \alpha(\boldsymbol{x}) vw.$$

We consider a discretization of the variational problem (2.1) with piecewise linear continuous finite elements. To define the finite element spaces and the approximate solution, we assume that we have a locally quasi–uniform, simplicial triangulation $\mathcal{T}_h$ of Ω. We assume that this triangulation also resolves $\mathcal{Y}_l$, namely, for $l = 1, \dots, m$ we have:

$$\bar{\Omega} = \cup_{\tau\in\mathcal{T}_h} \tau, \quad \bar{\mathcal{Y}}_l = \cup_{\tau\in\mathcal{T}_{Y,l}} \tau, \tag{2.4}$$

where $\mathcal{T}_{Y,l} \subset \mathcal{T}_h$, for $l = 1, \dots, m$. The standard space of piecewise linear (w.r.t $\mathcal{T}_h$) and continuous functions is denoted with V_h. The space of functions from V_h that vanish on the boundary of Ω is denoted with $V_{h,0}$.

Another notation that we frequently use is for functions in V_h restricted to a subdomain $D \subset \Omega$ that is resolved by $\mathcal{T}_h$. The space of restrictions of the functions from V_h on D is denoted by $V_h(D)$. The space of functions from V_h, which are supported in $\bar{D}$ is denoted with $V_{h,0}(D)$. Thus, $V_h(D) \subset H^1(D)$ and $V_{h,0}(D) \subset H_0^1(D)$ and we also frequently use the standard nodal value interpolation operators $I_h : C(\bar{D}) \mapsto V_h(D)$.

To finish this section let us write down the discrete problem that we want to solve: Find $u \in V_{h,0}$ such that

$$a(u, v) = (f, v), \quad \text{for all} \quad v \in V_{h,0}. \tag{2.5}$$

2.2. **Partition of unity.** In order to construct coarse spaces of practical interest we require some form of sparsity or localization. We will achieve this via partitions of unity subordinate to suitable overlapping partitions of the domain. We will construct a particular partition of unity whose elements are in V_h. The overlapping partition of the domain then simply consists of the interiors of the supports of the constructed partition of unity.

Consider a coarse quasi-uniform, simplicial triangulation $\mathcal{T}_H$, with characteristic mesh size H that covers Ω. We assume that the triangulation $\mathcal{T}_H$ covers $\overline{\Omega}$ exactly, although all the results below extend to the case when $\overline{\Omega} \subset \cup_{\tau\in\mathcal{T}_H} \tau$ with a strict inclusion. We do not require that $\mathcal{T}_H$ is in any way aligned with $\mathcal{T}_h$ (except on the boundary of Ω), neither do we assume that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ (although it could be). A natural partition of unity is provided by the canonical (nodal) basis functions for the piecewise linear (w.r.t. $\mathcal{T}_H$) continuous finite element space V_H. Let us denote these functions with $\{\chi_i^H\}_{i=1}^J$ and assume that they are ordered such that $\{\chi_i^H\}_{i=1}^{J_0}$ with $J_0 < J$ is the canonical basis for $V_{H,0}$. We then define

$$\chi_i := I_h(\chi_i^H), \quad \text{and} \quad \Omega_i := \text{interior}\,(\text{supp}(\chi_i)) \tag{2.6}$$

It is evident that for the domains Ω_i constructed in this way, there exists a fixed number N_0 such that for all $i = 1, \ldots, J$, we have

$$|N(i)| \leq N_0 \quad \text{where} \quad N(i) := \{j \mid \Omega_j \cap \Omega_i \neq \emptyset\}. \tag{2.7}$$

If $\mathcal{T}_h$ was a refinement of $\mathcal{T}_H$, then such a statement is clearly true, because we have that $\chi_i = \chi_i^H$ and $\mathcal{T}_H$ was assumed to be quasi-uniform. In the case when $\mathcal{T}_h$ is not a refinement of $\mathcal{T}_H$ the inequality in (2.7) is also easy to verify (see e.g. [26] for details). Our particular partition of unity $\{\chi_i\}_{i=1}^J$ has the following properties.

Lemma 2.1. *For $\{\chi_i\}_{i=1}^J$ and $\{\Omega_i\}_{i=1}^J$ as defined above, we have*

$$\sum_{i=1}^{J} \chi_i(\boldsymbol{x}) = 1 \quad \textit{and} \quad \bar{\Omega} = \bigcup_{i=1}^{J} \bar{\Omega}_i.$$

In addition, the following inequalities hold for $i = 1, \ldots, J$,

$$0 \leq \chi_i \leq 1, \quad \textit{and} \quad |\nabla \chi_i| \lesssim H_i^{-1}, \quad \textit{where} \quad H_i := \text{diam}(\Omega_i) \eqsim H.$$

Proof. We only prove the estimate on the gradient. All the other properties listed in the statement of the proposition are more or less obvious (they follow from similar properties of the functions $\{\chi_i^H\}_{i=1}^J$).

To bound the norm of the gradient, we commute the gradient with the nodal interpolation operator to obtain that

$$\nabla I_h(\chi_i^H) = I_{\mathcal{N}}(\nabla \chi_i^H). \tag{2.8}$$

Here $I_{\mathcal{N}}$ is the canonical interpolation from the lowest order Nédélec space on $\mathcal{T}_h$. We refer to [18, Theorem 3.1] for a proof of the commuting property (2.8) in the case of piecewise linear functions.

The degrees of freedom in the lowest order Nédélec space (associated with the edges E of $\mathcal{T}_h$) are $\frac{1}{|E|}\int_E \nabla \chi_i^H \cdot \tau_E$, which is clearly bounded by $\|\nabla \chi_i^H\|_\infty$. On the other hand, the Nédélec basis functions (dual to the degrees of freedom) have a constant L^∞-norm (independent of the mesh size) and so

$$\|\nabla \chi_i\|_\infty = \|I_{\mathcal{N}}(\nabla \chi_i^H)\|_\infty \lesssim \|\nabla \chi_i^H\|_\infty \lesssim H_i^{-1}$$

since $\mathcal{T}_H$ was assumed to be quasi-uniform. ☐

We now state the assumptions that link the distribution of the values of the coefficient $\alpha(\boldsymbol{x})$ of the PDE and the domain partitioning that we just introduced.

For a given $i \in \{1, \ldots, J\}$ let

$$M(i) := \{j \in \{1, \ldots, m\} | \Omega_i \cap \mathcal{Y}_j \neq \emptyset\}$$

and define $D_{il} := \Omega_i \cap \mathcal{Y}_l$, for all $l \in M(i)$. From the definition of D_{il} it follows that

$$\bar{\Omega}_i = \bigcup\nolimits_{l \in M(i)} \bar{D}_{il}. \tag{2.9}$$

Indeed,

$$\bigcup_{l\in M(i)} \bar{D}_{il} = \bigcup_{l\in M(i)} (\bar{\Omega}_i \cap \bar{\mathcal{Y}}_l) = \bar{\Omega}_i \cap \left(\bigcup_{l\in M(i)} \bar{\mathcal{Y}}_l \right) = \bar{\Omega}_i \cap \left(\bigcup_{l=1}^{m} \bar{\mathcal{Y}}_l \right) = \bar{\Omega}_i \cap \bar{\Omega} = \bar{\Omega}_i.$$

We make the following assumptions on the regions D_{il}, $i = 1, \ldots, J$, $l \in M(i)$:

C1. We assume that the cardinality of $M(i)$ is uniformly bounded by a constant m_0.

C2. We assume that $\mathcal{T}_h$ is sufficiently fine such that D_{il} contains at least one vertex from $\mathcal{T}_h$ in its interior, for all $i = 1, \ldots, J$ and $l \in M(i)$.

C3. For each domain D_{il}, $1 \le i \le J$, $l \in M(i)$, we assume that the following Poincaré inequality holds:

$$\inf_{c\in\mathbb{R}} \int_{D_{il}} (v-c)^2 \lesssim |D_{il}|^{2/d} \int_{D_{il}} |\nabla v|^2, \quad \text{for all} \quad v \in H^1(\mathcal{D}_{il}). \tag{2.10}$$

Remark 2.2. Basically, Assumption **C3** states that the Poincaré inequality (2.10) holds for each of the domains D_{il} with a constant $c_P^2(D_{il})$ that is proportional to $|D_{il}|^{2/d}$.

It is well known that the Poincaré constant $c_P(D)$ depends on the geometric characteristics of the domain D. For convex domains (see e.g. [20, 1, 2]) we have $c_P(D) \le \frac{\operatorname{diam}(D)}{\pi}$. If the domain D is not convex, then the dependence is more intricate. Following Cheeger [8], for the case that D is a John domain – which includes Lipschitz domains, star–shaped domains, and domains that have the cone property – it can be shown using the Sobolev-Poincaré inequality (see e.g. [19, 8]) that

$$c_P(D) \le 2c_I(D)|D|^{\frac{1}{d}}, \tag{2.11}$$

where $c_I(D)$ is the isoperimetric constant for D. It is scaling invariant and (in the case of connected polygonal/polyhedral domains) given by

$$c_I(D) = \sup_{S\subset D} \frac{\min\{|S|, |D\backslash S|\}^{(d-1)/d}}{|\partial S|}.$$

Thus, a sufficient condition for Assumption **C3** to hold is that the isoperimetric constant $c_I(D_{il})$ for each of the domains D_{il} is uniformly bounded. Note that without this assumption, in the worst case, $c_I(D_{il})$ may grow like $(H_i/h_i)^{d-1}$, where h_i is the characteristic mesh size of $\mathcal{T}_h$ on D_{il}, e.g. if D_{il} is hourglass–shaped.

We will also need the following Lemma on the stability of the nodal interpolant $\mathcal{I}_h$ on V_h.

Lemma 2.3. *Let u_q be a continuous, piecewise quadratic (w.r.t. $\mathcal{T}_h$) function. Then*

$$|I_h(u_q)|_a \lesssim |u_q|_a \quad \text{and} \quad \|I_h(u_q)\|_{0,\alpha} \lesssim \|u_q\|_{0,\alpha} \tag{2.12}$$

Proof. Let $\tau \in \mathcal{T}_h$ and let $E := (\boldsymbol{x}_i, \boldsymbol{x}_j)$ be the edge in τ with vertices $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Also, let $\partial_e u$ denote the directional derivative of u along E. We will provide a constructive proof of (2.12). To achieve this we use the following identity:

$$[I_h(u_q)](\boldsymbol{x}) = u_q(\boldsymbol{x}) - \frac{1}{2}\sum\nolimits_{E\in\tau} |E|^2 \partial_{ee} u \, \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{x}), \quad \text{for all} \;\; \boldsymbol{x} \in \tau,$$

where $\lambda_i(\boldsymbol{x})$ and $\lambda_j(\boldsymbol{x})$ are the barycentric coordinates corresponding to $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, resp.

To show the inequalities in (2.12) we proceed as follows: we apply the inverse inequality along each edge E, and use a well known formula for integrals of products of barycentric coordinates to obtain

$$|I_h(u_q)|_{H^1(\tau)} \lesssim |u_q|_{H^1(\tau)} \quad \text{and} \quad \|I_h(u_q)\|_{L_2(\tau)} \lesssim \|u_q\|_{L_2(\tau)}$$

The result follows, if we multiply each of the inequalities by $\alpha|\tau$ and sum over all $\tau \in \mathcal{T}_h$. $\square$

2.3. **Two–level multiplicative Schwarz algorithm.** To motivate the remainder of the paper, in this section we sketch the proof of a well known result on the convergence of the multiplicative Schwarz algorithm and highlight the key theoretical requirement on the coarse space that is needed to have a complete proof of uniform convergence, with rate independent of the coefficient variation.

Let $V_i = V_{h,0}(\Omega_i)$, $i = 1, \dots J$, and let V_0 be a "coarse" subspace of $V_{h,0}(\Omega)$ (unspecified for now). We denote with P_k the elliptic projection on V_k, defined as

$$a(P_k v, w_k) = a(v, w_k), \quad \text{for all} \quad w_k \in V_k, \quad k = 0, \dots, J.$$

For a given $f \in L_2(\Omega)$, the action $B_{\mathrm{MS}}^{-1} f$ of the two–level multiplicative Schwarz preconditioner $B_{\mathrm{MS}}^{-1} : V_{h,0} \mapsto V_{h,0}$ is obtained as follows

Algorithm 2.4 (Multiplicative Schwarz preconditioner)**.**
Let $f \in L_2(\Omega)$ be given. Set $u_{-J-1} = 0$.

for $k = -J : J$

Let $e_k \in V_{|k|}$ be the solution of

$$a(e_k, v_k) = (f, v_k) - a(u_{k-1}, v_k), \quad \textit{for all} \quad v_k \in V_{|k|}\,.$$

Define $u_k := u_{k-1} + e_k$.

endfor

Set $B_{\mathrm{MS}}^{-1} f = u_J$.

Since we consider here the multiplicative method, which is a convergent method, it is clear that $a(v, v) \le (B_{\mathrm{MS}} v, v)$. Henceforth, to estimate the convergence rate (or the condition number of the preconditioned system) we need an estimate of the form $(B_{\mathrm{MS}} v, v) \lesssim a(v, v)$. The following theorem is a classical result (see e.g. [32, 26, 29]).

Theorem 2.5. *Let us assume that for all $v \in V_{h,0}(\Omega)$, there exists $v_0 \in V_0$ such that*

$$\|v_0\|_a \lesssim \|v\|_a \quad \textit{and} \quad \|v - v_0\|_{0,\alpha} \lesssim H^2 \|v\|_a\,. \tag{2.13}$$

Then, $(B_{\mathrm{MS}} v, v) \lesssim a(v, v)$. The hidden constant depends on N_0 and the constants in (2.13).

Proof. We give a sketch of the proof to show that it is a direct consequence of (2.13). Note that $(B_{\mathrm{MS}} v, v)$ can be written as follows (see [29] or [9, Lemma 3.4]):

$$(B_{\mathrm{MS}} v, v) = \inf_{\sum v_i = v} \sum_{k=0}^{J} \left\| P_k \sum_{j=k}^{J} v_j \right\|_a^2. \tag{2.14}$$

This is sometimes referred to as the "XZ-identity" [32]. We now choose in particular the functions $v_i = I_h(\chi_i(v - v_0))$, for $i = 1, \dots, J$, and so $v = \sum_{i=0}^{J} v_i$. Setting $w = (v - v_0)$, it follows from (2.14) (by expanding the right hand side) that

$$(B_{\mathrm{MS}} v, v) \le \|v\|_a^2 + \sum_{k=0}^{J-1} \left\| P_k \sum_{j=k+1}^{J} v_j \right\|_a^2 \le \|v\|_a^2 + \|w\|_a^2 + \sum_{k=1}^{J-1} \left\| \sum_{j=k+1}^{J} \chi_j w \right\|_{a,\Omega_k}^2,$$

where in the last step we used Lemma 2.3 and the fact that $\|P_k\|_a = 1$, for all $k = 0, \dots, J$.

Since $\chi_j w|_{\Omega_k} \neq 0$ for at most N_0 values of j for each k, it follows from Lemma 2.1 and the triangle inequality that

$$\begin{aligned}(B_{\mathrm{MS}}v, v) &\lesssim \|v\|_a^2 + \|w\|_a^2 + \sum_{k=1}^{J} \|\chi_k w\|_a^2 \\ &\lesssim \|v\|_a^2 + \|v - v_0\|_a^2 + \sum_{k=1}^{J} \|\chi_k\|_\infty \|w\|_{a,\Omega_k}^2 + \|\nabla \chi_k\|_\infty \|w\|_{0,\alpha,\Omega_k}^2 \\ &\lesssim \|v\|_a^2 + \|v_0\|_a^2 + H^{-2}\|v - v_0\|_{0,\alpha}^2 ,\end{aligned}$$

and so it follows directly from (2.13) that $(B_{\mathrm{MS}}v, v) \lesssim a(v, v)$. □

As we can see, the key ingredient that is needed in the proof of Theorem 2.5 is a coarse space which has certain approximation and stability properties that hold regardless of the size of the coefficient variations. In §4 we will give a recipe to construct such coarse spaces based on energy minimization with constraints. The framework for these spaces is rather general and so we will first, in §3, prove an abstract approximation result. Concrete examples that fit into this general framework, based on solving local saddle point problems or local eigenvalue problems, are then discussed in §4.1 and in § 4.2, respectively.

3. An abstract approximation result

We consider the following variational problem: Find $u \in V$ such that

$$a(u, v) = f(v), \quad \text{for all} \quad v \in V. \tag{3.1}$$

Here $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ is a symmetric and continuous bilinear form, and $f \in V'$ is a continuous linear form. We make the following assumptions on V and $a(\cdot,\cdot)$:

A1. The bilinear form $a(\cdot,\cdot)$ is positive semi-definite and defines a semi-norm $|\cdot|_a$ on V, i.e.

$$|v|_a^2 = a(v, v) \geq 0, \quad \text{for all} \quad v \in V.$$

In addition, we assume that $V \subset \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space with a norm $\|\cdot\|$, and that for $v \in V$, the expression $\sqrt{\|v\|^2 + |v|_a^2}$ defines a norm on V.

A2. There exists a collection of linear functionals $\{f_l\}_{k=1}^m \subset V'$, with the following property: For every $\boldsymbol{q} \in \mathbb{R}^m$, there exists a $v_{\boldsymbol{q}} \in V$ such that

$$f_l(v_{\boldsymbol{q}}) = q_l, \quad \text{and} \quad \|v_{\boldsymbol{q}}\| \lesssim c_q \|\boldsymbol{q}\|_{l^2(\mathbb{R}^m)}.$$

A3. There are two constants c_a and c_f such that

$$\|v\|^2 \leq c_a |v|_a^2 + c_f \sum_{l=1}^{m} |f_l(v)|^2 , \quad \text{for all} \;\; v \in V. \tag{3.2}$$

We denote the subspace where $\{f_l\}_{l=1}^m$ vanish with Z. It is defined as

$$Z := \{v \in V \mid f_l(v) = 0, \quad l = 1, \ldots, m\} \tag{3.3}$$

Clearly, it follows from Assumptions **A1** and **A3** that the following variational problem has a unique solution: Find $u_0 \in Z$, such that

$$a(u_0, w) = g(w), \quad \text{for all} \quad w \in Z,$$

provided $g(\cdot)$ is a continuous linear form on Z.

3.1. **Stable (norm-1) projections via minimization.** In this section, we define a projection on a finite dimensional subspace $V_0 \subset V$, and this projection has appropriate stability and weak approximation properties. Consider the following constrained minimization problem: Given $\boldsymbol{q} \in \mathbb{R}^m$, find $u \in V$ such that

$$u = \arg\min_{v \in V} |v|_a^2, \quad \text{subject to the constraints} \quad f_l(u) = q_l, \quad l = 1, \dots, m. \tag{3.4}$$

The following lemma summarizes some of the properties of the minimizer u.

Lemma 3.1. *Assume that **A1**–**A3** are satisfied. Then the minimization problem* (3.4) *has a unique solution u. Moreover,*

$$a(u, w) = 0, \quad \textit{for all} \quad w \in Z. \tag{3.5}$$

Proof. By Assumption **A2** there exists u_1 such that $f_l(u_1) = q_l$, for $l = 1, \dots, m$. Let u_0 be the unique element of Z such that

$$a(u_0, w) = -a(u_1, w), \quad \text{for all} \quad w \in Z. \tag{3.6}$$

Note that the definition of u_0 implies that equation (3.5) is satisfied for $u = u_0 + u_1$. We aim to show now that $u = u_0 + u_1$ is also a minimizer of (3.4). We have

$$\begin{aligned} a(u, u) &= a(u_0 + u_1, u_0) + a(u_0 + u_1, u_1) \\ &= a(u_0, u_1) + a(u_1, u_1) \ = \ a(u_1, u_1) - a(u_0, u_0) \ \leq \ a(u_1, u_1), \end{aligned}$$

where we have used (3.5) to conclude $a(u_0 + u_1, u_0) = 0$. Since u_1 was arbitrary element of V satisfying the constraints, we may conclude that u is a minimizer.

To prove uniqueness, let $v_1 \in V$ be another element of V that satisfies the same constraints, that is, $f_l(v_1) = q_l$, and let v_0 be the solution to

$$a(v_0, w) = -a(v_1, w), \quad \text{for all} \quad w \in Z.$$

We would like to show that $u_1 + u_0 = v_1 + v_0$, which will imply that the solution is unique. Indeed, from the equations for u_0 and v_0 we get that

$$a(u_0 - v_0, w) = -a(u_1 - v_1, w), \quad \text{for all} \quad w \in Z,$$

Since $(u_0 - v_0) \in Z$, $(u_1 - v_1) \in Z$ and $a(\cdot, \cdot)$ is positive definite on Z, it follows that $u_0 - v_0 = -(u_1 - v_1)$, which is another way to say that $u_1 + u_0 = v_1 + v_0$. $\square$

Consider now m such minimization problems, whose solutions we denote with $\{\Phi_l\}_{l=1}^m$, such that

$$\Phi_l = \arg\min_{v \in V} |v|_a^2, \quad \text{subject to} \quad f_j(\Phi_l) = \delta_{jl} \quad l = 1, \dots, m. \tag{3.7}$$

From Lemma 3.1 it follows that each of the minimization problems in (3.7) is uniquely solvable. We now define $V_0 = \operatorname{span}\{\Phi_l\}_{l=1}^m$, which is clearly a finite dimensional space with $\dim V_0 \leq m$, as well as the projection $\Pi : V \mapsto V_0$, which for a given $v \in V$ is

$$\Pi v = \sum_{l=1}^m f_l(v) \Phi_l. \tag{3.8}$$

The following lemma shows that Πv also solves a minimization problem similar to (3.4).

Lemma 3.2. *For a given $u \in V$, let $\widetilde{u}$ be the solution to the following minimization problem.*

$$\widetilde{u} = \arg\min_{v\in V} |v|_a^2, \quad \textit{subject to} \quad f_l(\widetilde{u}) = f_l(u) \quad l = 1, \ldots, m. \tag{3.9}$$

Then $\widetilde{u} = \Pi u$, and so Πu defined in (3.8) *is the unique minimizer in* (3.9).

Proof. To check that Πu as defined in (3.8) satisfies the constraints is straightforward. We now set $\widetilde{u}_1 = \Pi u = \sum_{l=1}^m f_l(u)\Phi_l$, and proceeding as in the proof of Lemma 3.1 we have that $\widetilde{u} = \widetilde{u}_0 + \widetilde{u}_1$, where $\widetilde{u}_0 \in Z$ solves

$$a(\widetilde{u}_0, w) = -a(\widetilde{u}_1, w), \quad \text{for all} \quad w \in Z. \tag{3.10}$$

To finish the proof, we need to show that $\widetilde{u}_0 = 0$. Substituting the expansion of $\widetilde{u}_1$ we have

$$a(\widetilde{u}_1, w) = \sum_{l=1}^m f_l(u)a(\Phi_l, w), \quad \text{for all} \quad w \in Z. \tag{3.11}$$

Since (3.5) implies $a(\Phi_l, w) = 0$, for all $w \in Z$ and $l = 1, \ldots, m$, the right hand side of (3.11) is equal to zero. Thus, it follows from (3.10) that $\widetilde{u}_0 = 0$, since $a(\cdot, \cdot)$ is invertible on Z, which completes the proof. $\square$

We now state and prove the stability and approximation properties of the projection Π.

Theorem 3.3. *The following inequalities hold true for all $u \in V$:*

$$|\Pi u|_a \le |u|_a, \quad \textit{(stability estimate)}, \tag{3.12}$$

$$\|u - \Pi u\| \le \sqrt{c_a}|u|_a, \quad \textit{(weak approximation property)}. \tag{3.13}$$

(Note that these results do not depend on the size of the constants c_q and c_f in **A2** *and* **A3**.*)*

Proof. The first inequality is obvious, because as we have shown in Lemma 3.2, Πu is the minimizer in (3.9) and then by construction, the inequality (3.12) holds.

The weak approximation property (3.13) is obtained in a straightforward fashion by first using Assumption **A3**, then using the fact that $(v - \Pi v) \in Z$, and applying (3.12). We have,

$$\begin{aligned} \|v - \Pi v\|^2 &\le c_a|v - \Pi v|_a^2 + c_f \sum_{l=1}^m |f(v - \Pi v)|^2 \\ &= c_a|v - \Pi v|_a^2 \le c_a \, \|I - \Pi\| \, |v|_a^2 = c_a|v|_a^2 \,. \end{aligned}$$

In the last line we used a result of Kato (cf., e.g., [29]), i.e., that $||I - \Pi|| = ||\Pi\|$ which holds for any nontrivial projection Π and any inner-product norm $\|.\|$. $\square$

Remark 3.4. We would like to point out here a relation between Assumption **A3** above and a classical approximation result, known as Bramble–Hilbert Lemma [3, 4]. To show this, let us introduce the standard norm $\|u\|_{m,p,D}$ and seminorms $|u|_{k,p,D}$, $k = 0, \ldots, m$ on the Sobolev space $W^{m,p}(D)$ on a domain D, i.e.

$$\|u\|_{m,p,D}^p = \sum_{k=0}^m |u|_{k,p,D}^p, \quad \text{and} \quad |u|_{k,p,D}^p = \sum_{|\beta|=k} \int_D \left|\frac{\partial^\beta u}{\partial x^\beta}\right|^p,$$

Here the domain D has to satisfy some smoothness conditions, and to fix this, let us say that D is a union of domains star-shaped with respect to a ball.

If in Assumption **A3** we set: **(a)** $|u|_a = |u|_{m,p,D}$ and $\|u\| = \|u\|_{m,p,D}$; **(b)** $f_l(\cdot)$ to be the Hahn-Banach extensions on $W^{m,p}(D)$ of the dual basis for the space $\mathcal{P}_{m-1}$ (polynomials of degree less than or equal to $(m-1)$), with duality pairing given by the standard $L_2(D)$ inner product; and **(c)** set $\mathcal{H} = V = W^{m,p}(D)$. Then the inequality (3.2) in Assumption **A3** implies the Bramble-Hilbert Lemma. Indeed, taking $q_u \in \mathcal{P}_{m-1}$ to be the polynomial for which $f_l(u - q_u) = 0$, for $l = 1, \ldots, m$ and applying (3.2) to $(u - q_u)$ gives

$$\inf_{q \in \mathcal{P}_{m-1}} \|u - q\|_{m,p,\Omega} \leq \|u - q_u\|_{m,p,\Omega} \leq c_a |u - q_u|_{m,p,\Omega} = c_a |u|_{m,p,\Omega}.$$

This latter estimate is found in [3, Theorem 1] and [4, Theorem 1] and is referred to as the Bramble-Hilbert Lemma.

4. Local coarse space construction

We now use the abstract framework developed in §3 to construct coarse spaces of dimension $\leq m$ for the particular problem (2.5) with the stability and approximation properties needed in Theorem 2.5, i.e. satisfying (2.13).

The constructions and estimates from this section are applied locally to each of the subdomains Ω_i. Clearly Assumption **A1** holds on all of Ω, and thus also on each Ω_i. For certain choices of functionals $f_l(\cdot)$ we prove now that Assumptions **C1–C3** (in §2.2) imply Assumptions **A2** and **A3**.

To avoid a proliferation of indices in this section, let $i \in \{1, \ldots, J\}$ be fixed and set $D_l := D_{il}$ and $M = |M(i)|$. Thus, in particular, the relation (2.9) takes the form

$$\bar{\Omega}_i = \cup_{l=1}^{m} \bar{D}_l. \tag{4.1}$$

Also, let $\mathcal{H} := L_2(\Omega_i)$ and $\|v\| := \|v\|_{0,\alpha,\Omega_i}$ in §3.

4.1. Local coarse space construction via local saddle point problems.

Suppose that on the domain Ω_i the functionals $f_l(\cdot)$, $l = 1, \ldots, M$, are defined as

$$f_l(v) := \frac{1}{|D_l|} \int_{D_l} v. \tag{4.2}$$

To apply the abstract theory in §3 we now verify in turn Assumptions **A2** and **A3**.

Lemma 4.1. *Let $f_l(\cdot)$ be defined as in* (4.2). *Then Assumption **A2** holds true.*

Proof. Let $l \in \{1, \ldots, M\}$ be fixed. We first show that there exists a function $\psi_l \in V_{h,0}(D_l)$ and a constant c_l such that

$$\int_{D_l} \psi_l = |D_l| \quad \text{and} \quad \int_{D_l} \psi_l^2 \leq c_l |D_l|. \tag{4.3}$$

Let $\theta \in V_{h,0}(D_l)$ be such that $\theta(\boldsymbol{x}_j) = 1$ for every vertex $\boldsymbol{x}_j$ of $\mathcal{T}_h$ that is interior to D_l. The set of such vertices is denoted by $\mathcal{I}$. It follows from **C2** that $\mathcal{I} \neq \emptyset$. Let $\mathcal{T}_h(D_l) \subset \mathcal{T}_h$ be the restriction of $\mathcal{T}_h$ to D_l, and let $\omega_0 \subset D_l$ be the union of elements $\tau \in \mathcal{T}_h(D_l)$ that contain at least one interior vertex, i.e.

$$\omega_0 := \cup\{\tau \mid \tau \in \mathcal{T}_h(D_l) \text{ and } \tau \cap \mathcal{I} \neq \emptyset\}.$$

Integrating θ over D_l and using quadrature on each $\tau \in \mathcal{T}_h(D_l)$, we have

$$\int_{D_l} \theta = \sum_{\tau \in \mathcal{T}_h(D_l)} \int_\tau \theta = \frac{1}{d+1} \sum_{\tau \in \mathcal{T}_h(D_l)} |\tau| \sum_{\boldsymbol{x}_j^\tau \in \tau} \theta(\boldsymbol{x}_j^\tau) = \frac{1}{d+1} \sum_{\boldsymbol{x}_j \in \mathcal{I}} \sum_{\tau : \tau \supset \boldsymbol{x}_j} |\tau|,$$

where we denoted the vertices of τ with $\{\boldsymbol{x}_j^\tau\}_{j=1}^{d+1}$. Clearly, each of the elements $\tau \subset \omega_0$ appears at least once in the sum on the right side of the above relation and at most $(d+1)$ times (since each τ has $(d+1)$ vertices). None of the elements $\tau \subset D_l \backslash \omega_0$ appears. Hence,

$$\frac{1}{d+1}|\omega_0| = \frac{1}{d+1}\sum_{\tau\subset\omega_0}|\tau| \le \int_{D_l}\theta \le \sum_{\tau\subset\omega_0}|\tau| = |\omega_0|, \tag{4.4}$$

We now set

$$\psi_l := c_l\theta \quad \text{with} \quad c_l^{-1} := \frac{1}{|D_l|}\int_{D_l}\theta$$

It follows from (4.4) that $\frac{|D_l|}{|\omega_0|} \le c_l \le (d+1)\frac{|D_l|}{|\omega_0|}$. Thus, for each l we have defined ψ_l which by construction satisfies

$$\int_{D_l}\psi_l = |D_l| \quad \text{and} \quad \int_{D_l}\psi_l^2 = c_l^2\int_{D_l}\theta^2 \le c_l^2\int_{D_l}\theta \le c_l|D_l|.$$

Since $V_{h,0}(D_l) \subset V_h(\Omega_i)$ we have that $\psi_l \in V_h(\Omega_i)$.

To conclude the proof of **A2**, for a given $\boldsymbol{q} \in \mathbb{R}^M$ we set $v_{\boldsymbol{q}} = \sum_{l=1}^M q_l\psi_l$ and we obtain

$$\|u\|_{0,\alpha,\Omega_i}^2 = \sum_{l=1}^M q_l^2\int_{D_l}\alpha|\psi_l|^2 \lesssim \sum_{l=1}^M \alpha_l c_l|D_l|q_l^2 \le c_q\|\boldsymbol{q}\|_{\ell_2(\mathbb{R}^M)}^2 \quad \text{with} \quad c_q := \max_{l=1}^M \alpha_l c_l|D_l|.$$

(Recall that the constant c_q which depends on α and H does not appear in the stability and weak approximation bounds in Theorem 3.3.) □

Lemma 4.2. *Let $f_l(\cdot)$ be defined as in* (4.2). *Then Assumption **A3** holds true with constants $c_a \eqsim \max_{1\le l\le M}|D_l|^{2/d}$ and $c_f \eqsim \max_{1\le l\le M}\alpha_l|D_l|$.*

Proof. Because of Assumption **C3** we can apply the Poincaré inequality (2.10) on each of the domains D_l, multiply by α_l and sum over $l = 1,\dots,M$ to obtain,

$$\|u\|_{0,\alpha,\Omega_i}^2 \lesssim \sum_{l=1}^M |D_l|^{2/d}\int_{D_l}\alpha|\nabla u|^2 + \sum_{l=1}^M \alpha_l|D_l||f_l(u)|^2$$

which shows that **A3** holds true with

$$c_a \eqsim \max_{1\le l\le M}|D_l|^{2/d} \le |\Omega_i|^{2/d} \le H_i^2 \quad \text{and} \quad c_f \eqsim \max_{1\le l\le M}\alpha_l|D_l|.$$

□

Let us now show how we can use the results in §3 to construct a suitable coarse space on each of the subdomains Ω_i. We need to distinguish between subdomains Ω_i, $i \le J_0$, associated with interior coarse mesh vertices, and those with $i > J_0$, associated with coarse vertices on the (Dirichlet) boundary $\partial\Omega$ of the global domain.

Let us first consider Ω_i with $i \le J_0$. Having verified **A1**–**A3** for the functionals in (4.2), the M functions $\{\Phi_l\}_{l=1}^M$ in (3.7) span a coarse space on Ω_i with the desired properties. The corresponding projection operator given in (3.8), we denote here with Π_{Ω_i}. It follows from Lemmas 3.1 and 3.2 that in practice, this coarse space can be constructed by solving a family of saddle point problems on Ω_i.

From the proof of Lemma 4.1, we know that there exist functions ψ_l such that

$$f_j(\psi_l) = \delta_{jl}.$$

To state the relevant saddle point problems, we need the space of piecewise constant functions, with respect to the partitioning of Ω_i into the non-overlapping set of domains D_l. This space we denote with $W(\Omega_i)$, and define it formally as

$$W(\Omega_i) := \{q \in L^2(\Omega_i) \quad | \quad \sum_{l=1}^{M} q_l \mathbf{1}_{D_l}\},$$

where $\mathbf{1}_{D_l}$ is the characteristic function of D_l and $q_l \in \mathbb{R}$. We also introduce the local projection operator, $\widetilde{Q} : V_h(\Omega_i) \mapsto W(\Omega_i)$,

$$\widetilde{Q}v = \sum_{l=1}^{M} \left(\frac{1}{|D_l|} \int_{D_l} v \right) \mathbf{1}_{\mathcal{Y}_l}$$

The lth basis function Φ_l can now be computed by solving the following saddle point problem for $(\Phi_l, s) \in V_h(\Omega_i) \times W(\Omega_i)$:

$$\begin{array}{lllll} a(\Phi_l,\ v) & + & (\widetilde{Q}v, s)_{L_2(\Omega_i)} & = 0, & \text{for all } v \in V_h(\Omega_i), \\ (\widetilde{Q}\Phi_l, w)_{L_2(\Omega_i)} & & & = (\widetilde{Q}\psi_l, w)_{L_2(\Omega_i)}, & \text{for all } w \in W(\Omega_i). \end{array} \tag{4.5}$$

Note that the functions $\{\Phi_l\}_{l=1}^M$, constructed in this way, form a partition of unity over Ω_i. This immediately follows from the following: (1) $\Pi_{\Omega_i} 1 = 1$, since the constant function $1 \in V_h$ and minimizes $a(\cdot,\cdot)$ under the constraint $\mathbf{q} = \mathbf{1}$; and (2) $\widetilde{Q}(\sum_{l=1}^M \psi_l) = 1$ on Ω_i.

If $i > J_0$ we have to slightly modify the saddle point problems in (4.5) in order to obtain a global coarse space that is H_0^1–conforming, i.e. $V_0 \subset V_{h,0}$, later in §5. However, all we need to do is to replace the pure Neumann problems in (4.5) by problems with mixed Dirichlet/Neumann conditions by replacing

$$V_h(\Omega_i) \quad \text{with} \quad V_{h,0,\partial\Omega}(\Omega_i) := \{v_h \in V_h(\Omega_i) \mid v_h = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}.$$

Note that in this case the functions $\{\Phi_l\}_{l=1}^M$ do not form a partition of unity over Ω_i.

This is a fully local construction on each of the regions Ω_i. For each $i = 1, \ldots, J$, we need to solve $|M(i)| \le m_0$ of the saddle point problems (4.5), which is comparable in computational cost to the construction of the basis functions in multiscale finite elements (see e.g. [10]).

To conclude this section, we state the following theorem, which is a direct corollary from Theorem 3.3.

Theorem 4.3. *Let $u \in V_h(\Omega_i)$, for $i \le J_0$, or let $u \in V_{h,0,\partial\Omega}(\Omega_i)$, for $i \le J_0$, be arbitrary. Let $\Pi_{\Omega_i} u$ be its projection on the* span$\{\Phi_l\}_{l \in M(i)}$ *with Φ_l defined in (4.5), where*

$$\Pi_{\Omega_i} u := \sum_{l \in M(i)} f_l(u) \Phi_l \quad \textit{and} \quad f_l(u) := \frac{1}{|D_l|} \int_{D_l} u.$$

Then the following estimates hold:

$$|\Pi_{\Omega_i} u|_{a,\Omega_i} \le |u|_{a,\Omega_i}, \quad \textit{(stability estimate)}, \tag{4.6}$$

$$\|u - \Pi_{\Omega_i} u\|_{0,\alpha,\Omega_i} \lesssim |\Omega_i|^{1/d} |u|_{a,\Omega_i}, \quad \textit{(weak approximation property)}. \tag{4.7}$$

4.2. **Local coarse space construction via local eigenvalue problems.** Here we apply the abstract framework to coarse spaces constructed via local eigensolves, as proposed in [5, 7, 15, 13] within various contexts.

Instead of M saddle point problems, we consider the solution of the following eigenvalue problem on Ω_i: Find $\eta_l \in V_h(\Omega_i)$, and $\lambda_l \geq 0$, such that

$$a(\eta_l, w) = \lambda_l (\alpha \eta_l, w)_{L_2(\Omega_i)}, \quad \text{for all} \quad w \in V_h(\Omega_i). \tag{4.8}$$

Let us assume that the eigenvalues λ_l are ordered according to their size and that the eigenvectors are normalized such that $\|\eta_l\|_{0,\alpha,\Omega_i} = 1$. Then, given a constant $c_a > 0$, we define M here to be the largest integer such that $\lambda_{M+1}^{-1} \leq c_a$. The functionals $f_l(u)$ are chosen to be

$$f_l(u) = (\alpha \eta_l, u)_{L_2(\Omega_i)}, \quad \text{for all} \ \ l \leq M. \tag{4.9}$$

We note that since the set of eigenvectors $\{\eta_l\}$ forms a complete basis for $V_h(\Omega_i)$, we have

$$u = \sum\nolimits_l (\alpha \eta_l, u)_{L_2(\Omega_i)} \, \eta_l. \tag{4.10}$$

Again we need to replace $V_h(\Omega_i)$ by $V_{h,0,\partial\Omega}(\Omega_i)$, for $i > J_0$.

Lemma 4.4. *Let $f_l(\cdot)$ be defined as in* (4.9)*. Then Assumptions* ***A2*** *and* ***A3*** *hold true.*

Proof. Assumption **A2** is immediate. Given $\boldsymbol{q} \in \mathbb{R}^M$, choose $v_{\boldsymbol{q}} := \sum_{l=1}^M q_l \eta_l$. Then

$$f_l(v_{\boldsymbol{q}}) = (\alpha \eta_l, v_{\boldsymbol{q}})_{L_2(\Omega_i)} = q_l \quad \text{and} \quad \|v_{\boldsymbol{q}}\|_{0,\alpha,\Omega_i}^2 = \sum_{l=1}^M q_l^2 \|\eta_l\|_{0,\alpha,\Omega_i} = \|\boldsymbol{q}\|_{l^2(\mathbb{R}^M)}^2.$$

Proving Assumption **A3** is also straightforward. Using (4.8–4.10) and the definition of M

$$\begin{aligned}
\|u\|_{0,\alpha,\Omega_i}^2 &= \sum_{l \geq M} [(\alpha \eta_l, u)]^2 + \sum_{l \leq M} |(\alpha \eta_l, u)|^2 \\
&\leq \sum_{l \geq M} \frac{\lambda_l}{\lambda_{M+1}} [(\alpha \eta_l, u)]^2 + \sum_{l \leq M} |f_l(u)|^2 \\
&\leq c_a \sum_{l \geq 1} \lambda_l [(\alpha \eta_l, u)]^2 + \sum_{l \leq M} |f_l(u)|^2 \\
&= c_a \sum_{l \geq 1} a(\eta_l, u)(\alpha \eta_l, u) + \sum_{l \leq M} |f_l(u)|^2 = c_a |u|_a^2 + \sum_{l \leq M} |f_l(u)|^2.
\end{aligned}$$

Since by assumption Ω_i is shape regular, it can be shown (cf. [16, 31]) that $\lambda_{M+1}^{-1} \lesssim |\Omega_i|^{2/d}$ for $M \geq |M(i)|$. □

Thus we can again apply Theorem 3.3 to get stability and weak approximation in $\mathrm{span}\{\eta_l\}_{l=1}^M$.

Theorem 4.5. *The results of Theorem 4.3 remain true for*

$$\Pi_{\Omega_i} u := \sum_{l=1}^M f_l(u) \eta_l \quad \textit{and} \quad f_l(u) := (\alpha \eta_l, u)_{L_2(\Omega_i)}$$

and $\{\eta_l\}_{l=1}^M$ as defined in (4.9)*.*

5. Global stability and interpolation estimates

Here we put together the local constructions done in the previous section and construct a coarse space $V_0 \subset V_{h,0}$ on all of Ω using the partition of unity defined in Section 2.2. We concentrate only on the construction via the local saddle point problems described in §4.1. The construction via the local eigenproblems is identical and can be found (with proof) already in [15].

Following the construction in §4.1, for a given function $u \in V_{h,0}$ and a given domain Ω_i, $1 \le i \le J$, we consider as our functionals the averages

$$\bar{u}_{il} := \frac{1}{|D_{il}|} \int_{D_{il}} u, \quad \text{for all } \; l \in M(i),$$

and introduce the coarse grid interpolant of u to be the function u_0 defined as follows:

$$u_0 := I_h \left(\sum_{i=1}^{J} \chi_i \Pi_{\Omega_i} u \right), \quad \text{where} \quad \Pi_{\Omega_i} u := \sum_{l \in M(i)} \bar{u}_{il} \Phi_{il} \,, \tag{5.1}$$

$\{\chi_i\}_{i=1}^J$ is the partition of unity defined in (2.6) in §2.2 and the sets $\{\Phi_{il} : l \in M(i)\}$ contain the solutions to the $|M(i)|$ saddle point problems on Ω_i defined in (4.5) in §4.1. The corresponding coarse space is

$$V_0 := \mathrm{span}\{\Phi_{il}^H : 1 \le i \le J \text{ and } l \in M(i)\}, \quad \text{where} \quad \Phi_{il}^H := I_h(\chi_i \Phi_{il}).$$

The dimension of V_0 is $\sum_{i=1}^J |M(i)|$. Since by construction each of the functions $\Phi_{il} \in V_{h_0}$ and since $\mathcal{T}_H$ is aligned with the boundary of Ω, we have $V_0 \subset V_{h,0}$.

The following theorem shows that the interpolant u_0 defined above, which is an element of V_0, satisfies the stability and approximation properties (2.13) needed in Theorem 2.5.

Theorem 5.1. *Let $u \in V_{h,0}$ be given and let $u_0 \in V_0$ be the coarse grid interpolant of u, defined in* (5.1). *Then the following uniform estimates hold, with constants independent of h, H and the coefficient $\alpha(\boldsymbol{x})$:*

$$|u_0|_a^2 \;\lesssim\; |u|_a^2, \qquad \textit{(stability estimate)}, \tag{5.2}$$

$$\|u - u_0\|_{0,\alpha}^2 \;\lesssim\; H^2 |u|_a^2, \quad \textit{(weak approximation property)}. \tag{5.3}$$

Proof. It follows from the stability of I_h in Lemma 2.3 and the fact that $u = \sum_{i=1}^J \chi_i u$ that

$$\|u - u_0\|_{L_2(\mathcal{Y}_l)}^2 \;\lesssim\; \left\| u - \sum_{i=1}^J \chi_i \Pi_{\Omega_i} u \right\|_{L_2(\mathcal{Y}_l)}^2 = \left\| \sum_{i=1}^J \chi_i (u - \Pi_{\Omega_i} u) \right\|_{L_2(\mathcal{Y}_l)}^2 .$$

Now let $w_i := \chi_i(u - \Pi_{\Omega_i} u)$, multiply by α and sum over $l = 1, \ldots, m$, to get

$$\|u - u_0\|_{0,\alpha}^2 \lesssim \sum_{l=1}^m \sum_{i=1}^J \sum_{j=1}^J (w_i, w_j)_{0,\alpha,\mathcal{Y}_l}$$

Since w_i, and w_j are supported in Ω_i and Ω_j, respectively, changing the order of summation and applying the Cauchy-Schwarz inequality then gives

$$
\begin{aligned}
\|u-u_0\|_{0,\alpha}^2 &\lesssim \sum_{i=1}^{J} \sum_{j\in N(i)} \sum_{l\in M(i)\cap M(j)} (w_i, w_j)_{0,\alpha,\mathcal{Y}_l} \\
&\lesssim \sum_{i=1}^{J} \sum_{j\in N(i)} \sum_{l\in M(i)\cap M(j)} \|w_i\|_{0,\alpha,D_{il}} \|w_j\|_{0,\alpha,D_{jl}}.
\end{aligned}
\tag{5.4}
$$

Now we observe that $M(i)\cap M(j)$ is a subset of $M(i)$ and of $M(j)$, and so by applying the Cauchy-Schwarz inequality to the innermost sum we arrive at

$$
\sum_{l\in M(i)\cap M(j)} \|w_i\|_{0,\alpha,D_{il}} \|w_j\|_{0,\alpha,D_{jl}} \le \Big(\sum_{l\in M(i)} \|w_i\|_{0,\alpha,D_{il}}^2\Big)^{1/2} \Big(\sum_{l\in M(j)} \|w_j\|_{0,\alpha,D_{jl}}^2\Big)^{1/2}
$$

Substituting this in (5.4) and using the fact that the cardinality of $N(i)$ cannot exceed N_0 we finally obtain

$$
\begin{aligned}
\|u-u_0\|_{0,\alpha}^2 &\lesssim \sum_{i=1}^{J} \sum_{j\in N(i)} \|w_i\|_{0,\alpha,\Omega_i}\ \|w_j\|_{0,\alpha,\Omega_j} \\
&\lesssim \sum_{i=1}^{J} \sum_{j\in N(i)} \|w_i\|_{0,\alpha,\Omega_i}^2 + \|w_j\|_{0,\alpha,\Omega_j}^2 \lesssim \sum_{i=1}^{J} \|w_i\|_{0,\alpha,\Omega_i}^2
\end{aligned}
\tag{5.5}
$$

Since $\|\chi_i\|_\infty \lesssim 1$ by Lemma 2.1 we can now apply the weak approximation property (4.7) in Theorem 4.3 on each of the regions Ω_i to complete the proof of (5.3):

$$
\|u-u_0\|_{0,\alpha}^2 \lesssim \sum_{i=1}^{J} \|u-\Pi_{\Omega_i}u\|_{0,\alpha,\Omega_i}^2 \lesssim \sum_{i=1}^{J} |\Omega_i|^{2/d} \|u\|_{a,\Omega_i}^2 \lesssim H^2\|u\|_a^2
$$

The stability property can be proved in a similar fashion. As in (5.5) we can show that

$$
\|u-u_0\|_a^2 \lesssim \sum_{i=1}^{J} \|\chi_i(u-\Pi_{\Omega_i}u)\|_{a,\Omega_i}.
$$

Applying the triangle inequality and the product rule we get

$$
\|u_0\|_a^2 \lesssim \|u\|_a^2 + \sum_{i=1}^{J} \|\chi_i\|_\infty^2 \left(\|u\|_{a,\Omega_i}^2 + \|\Pi_{\Omega_i}u)\|_{a,\Omega_i}^2\right) + \|\nabla\chi_i\|_\infty^2 \|u-\Pi_{\Omega_i}u\|_{0,\alpha,\Omega_i}^2
$$

The stability property (5.2) then follows immediately from Lemma 2.1 and Theorem 4.3. □

Moving away from our motivating application of the new coarse spaces in the multiplicative Schwarz method, Theorem 5.1 also shows that V_0, constructed in this way, has optimal (in H) approximation properties (independent of the coefficient variation) in the weighted L_2–norm, which may be of interest e.g. for numerical upscaling.

6. Reducing the dimension via weighted Poincaré inequalities and non-constant coefficients

It is possible and straightforward now to considerably reduce the dimension of the coarse space V_0 constructed in the previous section, while still maintaining the stability and weak approximation properties (5.2) and (5.3), by resorting to the weighted Poincaré inequalities proved in [22, 21] instead of the standard Poincaré inequality. In addition this extends the theory to general, non piecewise constant coefficients α.

For any $i = 1, \ldots, J$, let $\{D_{il}\}_{l \in M(i)}$ be an arbitrary non-overlapping partitioning of Ω_i into polygonal/polyhedral domains (assumed to be resolved by $\mathcal{T}_h$) such that (2.9) holds and Assumptions **C1** and **C2** are satisfied. Most importantly, we do not assume any longer that the coefficient α is constant on D_{il}.

Note now that all the results that we proved in Sections 4.1 and 5 hold true, if we replace Assumption **C3** with the following assumption:

C3'. For each domain D_{il}, $1 \le i \le J$, $l \in M(i)$, we assume that the following weighted Poincaré inequality holds:

$$\inf_{c \in \mathbb{R}} \int_{D_{il}} \alpha (v - c)^2 \;\lesssim\; |D_{il}|^{2/d} \int_{D_{il}} \alpha |\nabla v|^2, \quad \text{for all} \quad v \in H^1(\mathcal{D}_{il}). \tag{6.1}$$

The only thing that changes are the constants c_q and c_f in Lemmas 4.1 and 4.2. Since the infimum in (6.1) is attained for $c_l^* = \int_{D_{il}} \alpha v / \int_{D_{il}} \alpha$, we get

$$c_q = \max_{l \in M(i)} c_l \int_{D_{il}} \alpha(\boldsymbol{x}) \quad \text{and} \quad c_f = \max_{l \in M(i)} |D_{il}| \max_{\boldsymbol{x} \in D_{il}} \alpha(\boldsymbol{x})$$

with c_l as defined in the proof of Lemma 4.1. Note that c_f can be reduced to $\max_{l \in M(i)} \int_{D_{il}} \alpha(\boldsymbol{x})$ if the functionals $f_l(u)$ in (4.2) are replaced by the weighted averages $f_l(u) = c_l^*$, but this has no bearing on the stability and weak approximation properties of V_0.

As shown in [22, 21] the key concept for **C3'** to hold with a constant that is independent of the coefficient variation within each of the regions D_{il} is that $\alpha(\boldsymbol{x})$ is locally quasi-monotone. To give some more details, let us fix $1 \le i \le J$ and $l \in M(i)$ and set $\omega := D_{il}$. We need to define the following subsets of ω where $\alpha(\boldsymbol{x})$ is constant:

$$\omega^k = \omega \cap \mathcal{Y}_k, \qquad \text{where} \quad k \in \mathcal{I}(\omega) := \{k : \omega \cap \mathcal{Y}_k \neq \emptyset\}.$$

Let us assume w.l.o.g. that each of these subregions is connected. The following slightly generalizes the notion of quasi-monotonicity coined in [11]. Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed combinatorial graph with $\mathcal{N} = \{\omega^k : k \in \mathcal{I}(\omega)\}$. The edges are ordered pairs of vertices defined as follows.

Definition 6.1. *Suppose that $\gamma^{k,k'} = \overline{\omega}^k \cup \overline{\omega}^{k'}$ is a sufficiently regular, non-empty manifold of dimension $d-1$ such that $\operatorname{meas}(\gamma^{k,k'}) \eqsim \operatorname{meas}(\omega^k \cup \omega^{k'})^{2/d}$. The ordered pair $(\omega^k, \omega^{k'})$ is an edge in $\mathcal{E}$, if and only if $\alpha_k \lesssim \alpha_{k'}$.*

Quasi-monotonicity is related to the connectivity in this graph. Let $k^* \in \mathcal{I}(\omega)$ be the index of the region ω^k with the largest coefficient, i.e. $\alpha_{k^*} = \max_{k \in \mathcal{I}(\omega)} \alpha_k$.

Definition 6.2. *The coefficient α is quasi-monotone on $\omega = D_{il}$, if there is a path in $\mathcal{G}$ from any vertex ω^k to ω^{k^*}.*

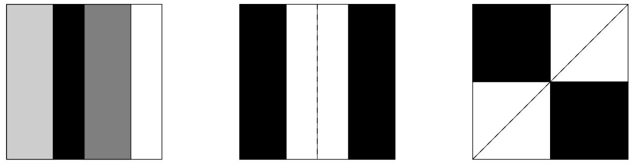


Figure 6.1: Example of a quasi-monotone coefficient in (a) and two typical examples of non-quasi-monotone coefficients in (b-c). Darker colours indicate a larger coefficient. The dashed line indicates a possible partitioning so that the coefficient is quasi–monotone in each of the subregions.

The coefficient in Figure 6.1(a) is an example of a quasi-monotone coefficient. The coefficients in Figure 6.1(b-c) are not quasi-monotone. The following lemma is proved in [22, 21].

Lemma 6.3. *If α is quasi-monotone on D_{il}, for all $1 \le i \le J$ and $l \in M(i)$, then Assumption **C3'** holds with hidden constant independent of α.*

Note that a similar result can be proved in the case of non piecewise constant coefficients $\alpha(\boldsymbol{x})$, provided again that α is quasi-monotone (as defined in [22, 21] for the general case) on each of the regions D_{il}.

7. Numerical results

In this section, we demonstrate the performance of the two-level overlapping Schwarz method with coarse space constructed by solving local constrained minimization (saddle–point) problems (constrained energy min AMGe, for short) combined with partition of unity (motivated by Theorem 2.5 and based on the result in Theorem 5.1).

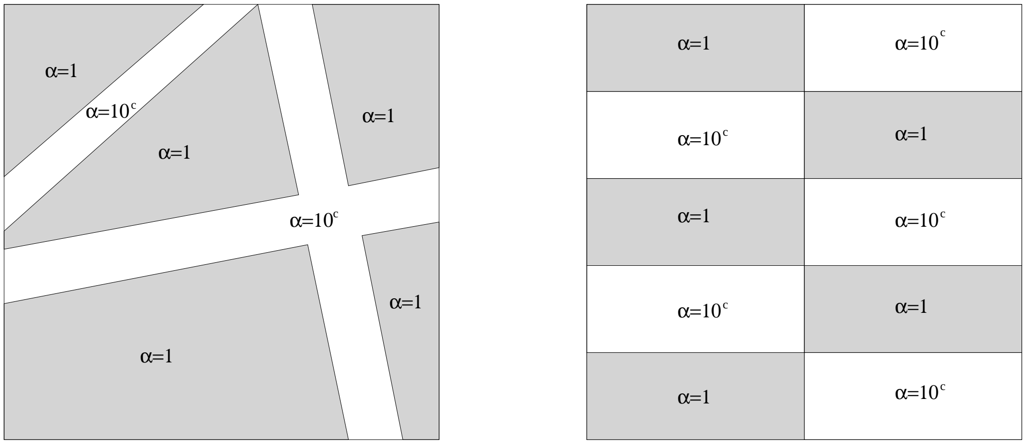
We consider two test problems with large jumps of coefficients and study the convergence of the method by varying the size of the jump (referred to as contrast). We also vary the fine-grid mesh size as well as the coarsening factor (ratio of the number of coarse-grid elements to the number of fine-grid ones). The coarse elements are obtained by agglomerating fine grid elements; so in general they are polygonal subdomains. As partition of unity, we use a trace minimization construction as described, for example, in [29] (or [33]).

The test problem we consider is the variational problem (2.1) on the unit square $\Omega \subset \mathbb{R}^2$, $\Omega = (0,1) \times (0,1)$, with $f(\boldsymbol{x}) = -1$. We restate this problem here for convenience: Find $u^* \in H_0^1(\Omega)$, such that

$$\int_\Omega \alpha(\boldsymbol{x})\ \nabla u^* \cdot \nabla v \ = - \int_\Omega v(\boldsymbol{x}) \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{7.1}$$

We consider two examples which have different coefficient distribution. In both cases the coefficient $a(\boldsymbol{x})$ is piecewise constant and takes two values, 1 and 10^c, where c (referred to as contrast) varies between -12 and 12. The coefficient distributions for Example 1 and Example 2 are shown in Figure 7.1.

Example 1. The first test problem corresponds to a subdomain that resembles the number "four" (see Figure 7.1a). Outside this subdomain the value of the coefficient is one. We use a locally refined mesh to resolve the coefficient only on the finest mesh. The coefficient $\alpha(\boldsymbol{x})$ for this example, as well as the finest grid (resolving the jumps) are shown in Figure 7.2.



(a) Example 1: $\alpha(\boldsymbol{x})$ distribution (b) Example 2: $\alpha(\boldsymbol{x})$ distribution

Figure 7.1: Domain and coefficient values for Example 1 and Example 2.

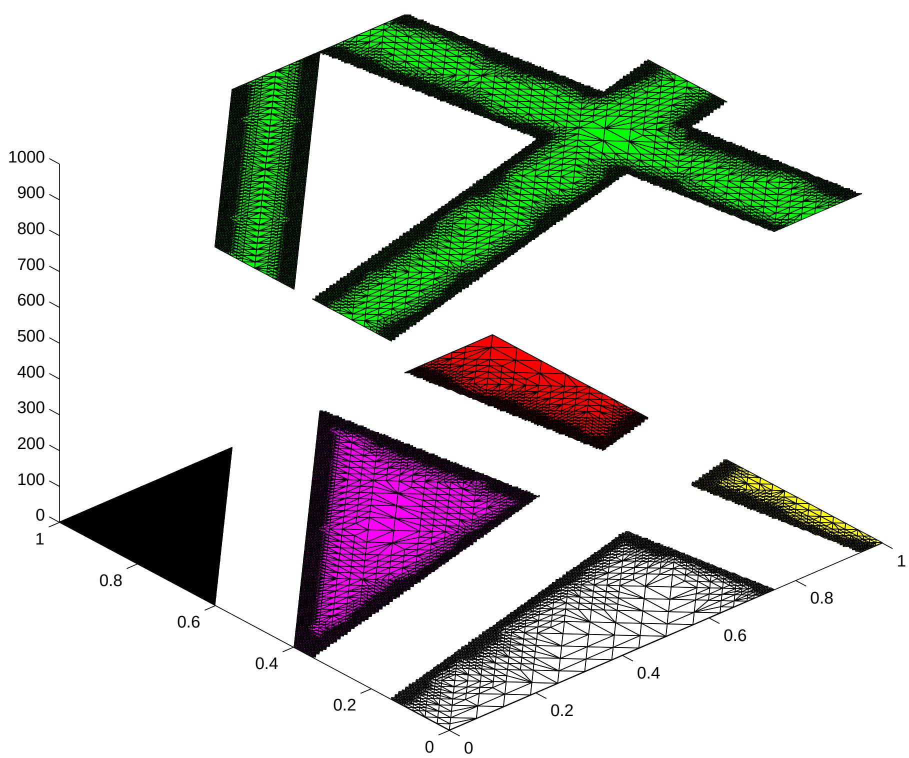


Figure 7.2: Discontinuous coefficient distribution and locally adapted mesh resolving it in Example 1.

Example 2. In the second problem the two constant values of the coefficient $\alpha(\boldsymbol{x})$ alternate within the domain, as shown in Figure 7.1b. The fine mesh for this problem is obtained by locally adapting an initially unstructured mesh that does not resolve the discontinuous coefficient in such a way that the final mesh does. This is illustrated in Figure 7.3.

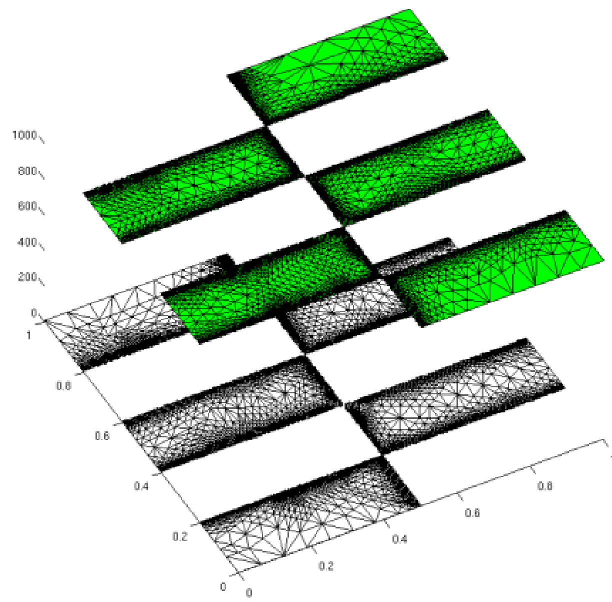


Figure 7.3: Fine grid resolved discontinuous coefficient in Example 2.

Convergence rates. In our tests we use the two–level Schwarz method as a stationary iteration method. A two level multiplicative Schwarz iteration is performed in the following way: (1) forward Schwarz loop over the subdomains; (2) coarse–grid correction; and (3) backward Schwarz loop over the subdomains. The coarse problem is solved by a preconditioned conjugate gradient method (up to a relative accuracy of 10^{-6}). The saddle point problems arising in the construction of the coarse space are solved by a direct method. The iterations are stopped when the ℓ_2–norm of the preconditioned residual is reduced by a factor of 10^{-6}. The subdomains in the Schwarz method are formed by putting together (in one subdomain) all agglomerated elements that have a common vertex. The agglomerated elements are obtained using the algorithm described in [29] (or more recently in [30]), and for Example 1 one of the coarse grids is shown in Figure 7.4. We point out here that according to the theory developed in the previous sections, the coarse grid agglomerated elements need not (and do not) resolve the coefficient jumps. This is clearly seen in Figure 7.4.

The performance of the two-level Schwarz method is summarized in Table 7.1 and Table 7.2 for the two examples, respectively. Although, strictly speaking, our theoretical results do not apply to the case of algebraically constructed coarse elements and partitions of unity using element agglomeration and energy minimization, the two-level Schwarz method, as seen in both tables (Table 7.1 and Table 7.2), exhibits convergence factors (ϱ) that are insensitive to variations in the contrast as well as the grid size. The convergence factors are also fairly insensitive with respect to the coarsening factor; compare Table 7.2 and Table 7.3.

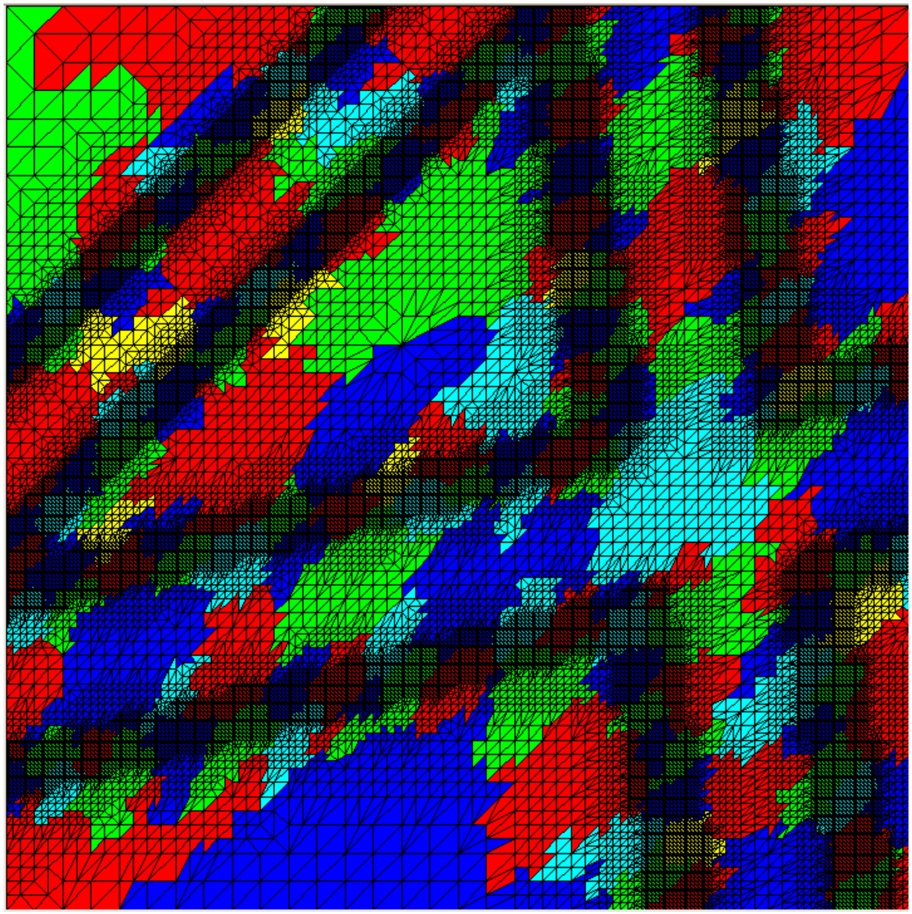


Figure 7.4: Coarse agglomerated elements for mesh of Example 1.

c	# coarse grid dofs	n_{it}	ϱ
-12	3,621	68	0.74
-9	3,638	66	0.75
-6	3,671	54	0.74
-3	3,724	53	0.75
0	3,762	43	0.72
3	3,723	52	0.75
6	3,690	56	0.75
9	3,626	65	0.75
12	3,574	73	0.74

Table 7.1: Convergence factor (ϱ) for the two–level Schwarz method with constrained energy min AMGe coarse space for Example 1. The fine (triangular) mesh is fixed with $192,892$ elements and $97,004$ dofs. The jump in the PDE coefficient is 10^c. The coarse mesh is chosen so that the coarsening factor $(\frac{\#\text{ fine-grid elements}}{\#\text{ coarse-grid elements}})$ is approximately 36.

# fine grid dofs	# fine grid elements	# coarse dofs	# coarse elements	n_{it}	ϱ
55,823	111,160	8,808	3,041	59	0.72
12,1072	241,420	18,477	6,464	55	0.70
144,785	288,448	18,944	6,985	89	0.79
184,669	367,776	21,164	9,103	70	0.73
271,681	542,160	38,521	14,330	84	0.78
325,345	649,008	41,149	16,388	77	0.77

Table 7.2: Convergence factor (ϱ) for the two–level Schwarz method with constrained energy min AMGe coarse space for Example 2. Fixed jump of 10^{12} in the PDE coefficient and variable fine-grid mesh. The coarse mesh is chosen so that the coarsening factor $(\frac{\text{\# fine-grid elements}}{\text{\# coarse-grid elements}})$ is approximately 36.

# fine grid dofs	# fine grid elements	# coarse dofs	# coarse elements	n_{it}	ϱ
58,536	116,428	12,832	8,244	55	0.71
118,196	235,868	17,292	13,475	56	0.70
144,785	288,488	29,730	19,826	56	0.70
272,140	543,195	42,805	36,108	64	0.71

Table 7.3: Convergence factor (ϱ) for the two–level Schwarz method with constrained energy min AMGe coarse space for Example 2. Fixed jump of 10^{12} in the PDE coefficient and variable fine-grid mesh. The coarse mesh is chosen so that the coarsening factor $(\frac{\text{\# fine-grid elements}}{\text{\# coarse-grid elements}})$ is approximately 16.

In summary, all experiments agree with our theoretical results given in the previous sections.

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