The parabolic Anderson model with heavy-tailed potential

Peter Mörters

joint work with

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The project

Aim: Study diffusion in a random medium or potential.

Questions:

• Which qualitative effects can be caused by small inhomogeneities in the medium?

• Which qualitative effects can be caused by considerable irregularity of the medium?
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This talk will focus on the second question, but we will start with a general introduction of the parabolic Anderson model.
The parabolic Anderson problem

The parabolic Anderson problem is the Cauchy problem for the heat equation

\[
\frac{\partial}{\partial t} u(t, z) = \Delta u(t, z) + \xi(z) u(t, z), \quad \text{for } (t, z) \in [0, \infty) \times \mathbb{Z}^d,
\]

\[
u(0, z) = 1_{0}(z), \quad \text{for } z \in \mathbb{Z}^d,
\]

with discrete Laplacian \((\Delta f)(z) = \sum_{y \sim z}[f(y) - f(z)]\) and random potential \(\{\xi(z): z \in \mathbb{Z}^d\}\) independent, identically distributed.
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with

- discrete Laplacian \((\Delta f)(z) = \sum_{y \sim z} [f(y) - f(z)]\)
- random potential \(\{\xi(z): z \in \mathbb{Z}^d\}\) independent, identically distributed.

The problem has a unique nonnegative solution if

$$E[(\xi(0) \vee 0)^{d+\varepsilon}] < \infty$$

for some \(\varepsilon > 0\), which will always be assumed.
Random mass transport in random media

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- Start a particle of mass one at the origin of $\mathbb{Z}^d$,
- suppose this particle performs a continuous time random walk $(X_s : s \geq 0)$ with generator $\Delta$, 

This is the content of the celebrated Feynman–Kac formula: 

$$u(t, z) = \mathbb{E}_0\{1_{X_t = z} \exp\left(\int_0^t \xi(X_s) \, ds\right)\}$$

for $t > 0$, $z \in \mathbb{Z}^d$. 
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Intermittency effect

For any nondegenerate potential distribution, the parabolic Anderson model is believed to exhibit an intermittency effect:

As time progresses, the bulk of the mass of the solution is not spreading in a regular fashion, but becomes concentrated in a small number of spatially separated islands of moderate size determined by the potential.
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Heuristics: In the Feynman-Kac formula

\[ \sum_{z \in \mathbb{Z}^d} u(t, z) = \mathbb{E}_0 \left\{ \exp \left( \int_0^t \xi(X_s) \, ds \right) \right\}. \]

there is a competition between the benefits of spending much time at sites with large potential values and the unlikeliness of this behaviour. The paths \((X_s : 0 \leq s \leq t)\) that give the dominant contribution to the integral are likely to end in certain regions of the lattice, the islands.
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Main contributors in this research area: Molchanov, Gärtner, König, Sznitman, den Hollander, . . . but there are still many open problems.
Rough classification of potentials

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In this talk we focus on a case of heavy tails and derive fine properties of the solution, including a detailed discussion of the number of islands in which the solution is concentrated.
Heavy tailed potentials

We now assume that $\xi(0)$ is Pareto-distributed, i.e.

$$\mathbb{P}\{\xi(0) \geq x\} = x^{-\alpha} \quad \text{for } x \geq 1,$$

so that $\xi(0)$ has a polynomial tail with parameter $\alpha > d$. 

Advantage: the intermittency effect is expected to be strongest with only a small number of islands consisting of single sites.

Disadvantage: Moments of the solution do not exist and new techniques have to be developed to study the problem.

Questions:

- How many sites are needed to support the bulk of the solution?
- Where are these sites?
- How fast does the solution grow?
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Complete localisation

**Theorem 1** *(König, Lacoin, M, Sidorova 2006)*

There exists a stochastic process \((Z_t: t > 0)\) with values in \(\mathbb{Z}^d\) such that

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\lim_{t \to \infty} \frac{u(t, Z_t)}{\sum_{z \in \mathbb{Z}^d} u(t, z)} = 1 \quad \text{in probability.}
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Remarks:

- The mass is concentrated in just one site, a phenomenon often called complete localisation. This has not been observed in any lattice-based model of mathematical physics so far.
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- The mass is concentrated in the maximiser \(Z_t\) of

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\Psi_t(z) = \xi(z) - \frac{\|z\|}{t} \log \frac{\|z\|}{2 \cdot \det}.
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Limit law for the concentration site and mass

Let $U(t) = \sum_{x \in \mathbb{Z}^d} u(t, x)$ be the total mass of the solution.
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**Theorem 2 (M, Ortgiese, Sidorova 2009)**

As \( t \to \infty \),

\[
\left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha - d}} Z_{st}, \left( \frac{\log t}{t} \right)^{\frac{d}{\alpha - d}} \frac{\log U(st)}{st} : s > 0 \right)
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\[\Rightarrow (Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha - d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0),\]

in the Skorokhod topology on every compact subinterval of \((0, \infty)\).
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Extreme value theory approach

Recall that

\[ \frac{1}{t} \log U(t) \approx \max_{z \in \mathbb{Z}^d} \Psi_t(z) \]

for

\[ \Psi_t(z) = \xi(z) - \frac{\|z\|}{t} \log \frac{\|z\|}{2 \det}. \]
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For \( r_t = (t / \log t)^{\frac{\alpha}{\alpha-d}} \) and \( a_t = (t / \log t)^{\frac{d}{\alpha-d}} \) the point process

\[
\Pi_t = \sum_{z \in \mathbb{Z}^d} \delta \left( \frac{z}{r_t}, \frac{\psi_t(x)}{a_t} \right)
\]

converges to a Poisson process with intensity measure

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\nu(dx \ dy) = dx \otimes \frac{\alpha dy}{(y + \frac{d}{\alpha-d}\|x\|)^{\alpha+1}}.
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\[ \nu(dx \, dy) = dx \otimes \frac{\alpha \, dy}{\left( y + \frac{d}{\alpha - d} \|x\| \right)^{\alpha + 1}}. \]

For fixed \( s \) and large \( t \) we obtain

\[ \frac{\Psi_{st}(z)}{a_t} \approx \frac{\Psi_t(z)}{a_t} + \frac{d}{\alpha - d} \left( 1 - \frac{1}{s} \right) \frac{\|z\|}{r_t} \cdot \]
Definition of the limit process

Let $\Pi$ be a Poisson point process with intensity measure

$$\nu(dx \, dy) = dx \otimes \frac{\alpha \, dy}{(y + \frac{d}{\alpha-d} \|x\|)^{\alpha+1}}.$$
Definition of the limit process

For $z > 0$ consider the cone

$\{(x, y) : y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}$. 

\[ -\frac{d}{\alpha-d}(1 - 1/t) \] 

\[ -\frac{d}{\alpha-d}|x| \]
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Let \( Y_s = (Y_s^{(1)}, Y_s^{(2)}) \) be the first point of \( \Pi \) hit by the cone as we decrease \( z \).
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![Diagram of a cone in a coordinate system with points scattered around it.]
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![Diagram showing the cone and the first point hit by the cone as $z$ decreases.](image)
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![Diagram of the cone and point](image-url)
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![Diagram showing the cone and the point of intersection]
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\[\text{\includegraphics{diagram.png}}\]
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\begin{tikzpicture}
    % Place the diagram content here.
\end{tikzpicture}
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Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 
Definition of the limit process

For $z > 0$ consider the cone

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![Diagram of a cone and points](image-url)
Definition of the limit process

For $z > 0$ consider the cone

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Let $Y_s = (Y^{(1)}_s, Y^{(2)}_s)$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of a cone with points distributed along the edges and the origin marked with a red dot.](image-url)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha - d} \left(1 - \frac{1}{s}\right) \|x\|\}.$$  

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y) : y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y^{(1)}_s, Y^{(2)}_s)$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of a cone and point of intersection]
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|}\.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of the cone and point](image-url)
Definition of the limit process

For $z > 0$ consider the cone 

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

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Peter Mörters (Bath)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of the cone and point](image)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha - d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

\[
\begin{align*}
\end{align*}
\]
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha - d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z.$
**Definition of the limit process**

For \( z > 0 \) consider the cone

\[
\{(x, y) : y \geq z - \frac{d}{\alpha - d} (1 - \frac{1}{s}) \|x\|\}.
\]

Let \( Y_s = (Y_s^{(1)}, Y_s^{(2)}) \) be the first point of \( \Pi \) hit by the cone as we decrease \( z \).
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

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Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram showing the cone and the first point hit by the cone](image-url)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha - d} (1 - \frac{1}{s}) ||x||\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

\begin{tikzpicture}
\draw[->] (-5,0) -- (5,0) node[right] {$x$};
\draw[->] (0,-5) -- (0,5) node[above] {$y$};
\draw[dashed] (-5,-5) -- (5,5);
\draw[dashed] (-5,5) -- (5,-5);
\draw (-5,0) -- (0,5);
\draw (0,5) -- (5,0);
\filldraw[red] (1,2) circle (2pt);
\end{tikzpicture}
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram showing the cone and the point of intersection](image)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of a cone with points scattered around it, indicating the region defined by the inequality above. The diagram shows a plane intersecting the cone at a specific point, marked with a red dot.]
Definition of the limit process

For \( z > 0 \) consider the cone

\[ \{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}. \]

Let \( Y_s = (Y_s^{(1)}, Y_s^{(2)}) \) be the first point of \( \Pi \) hit by the cone as we decrease \( z \).
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d} (1 - \frac{1}{s}) \|x\|\}.$$ 

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Definition of the limit process

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Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of a cone and a point hit by the cone](image-url)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y) : y \geq z - \frac{d}{\alpha - d} (1 - \frac{1}{s}) \|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 
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Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram](image-url)
Definition of the limit process
For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

\[ \begin{array}{c}
\text{y} \\
\text{x}
\end{array} \] 

\[ \begin{array}{c}
\text{y} \\
\text{x}
\end{array} \]
Definition of the limit process
For \( z > 0 \) consider the cone

\[
\{(x, y) : y \geq z - \frac{d}{\alpha - d} (1 - \frac{1}{s}) \|x\| \}. 
\]

Let \( Y_s = (Y_s^{(1)}, Y_s^{(2)}) \) be the first point of \( \Pi \) hit by the cone as we decrease \( z \).
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of a cone in a coordinate system with a point marked on the cone]

Peter Mörters (Bath)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\|\}.$$ 

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The parabolic Anderson model

Peter Mörters (Bath)
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha - d}(1 - \frac{1}{s})\|x\|\}.$$ 

Let $Y_s = (Y^{(1)}_s, Y^{(2)}_s)$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 
Definition of the limit process

For $z > 0$ consider the cone

$$\{(x, y): y \geq z - \frac{d}{\alpha-d}(1 - \frac{1}{s})\|x\| \}.$$ 

Let $Y_s = (Y_s^{(1)}, Y_s^{(2)})$ be the first point of $\Pi$ hit by the cone as we decrease $z$. 

![Diagram of the cone and point](image-url)
Definition of the limit process

\[
\left( \left( \frac{\log t}{t} \right)^\frac{\alpha}{\alpha-d} Z_{st}, \left( \frac{\log t}{t} \right)^\frac{d}{\alpha-d} \frac{\log U(st)}{st} : s > 0 \right)
\]

\[
\Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0 \right).
\]

The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
Definition of the limit process

\[
\left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha - d}} Z_{st}, \left( \frac{\log t}{t} \right)^{\frac{d}{\alpha - d}} \log U(st) \right) : s > 0
\]

\[ \Rightarrow (Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha - d} (1 - \frac{1}{s}) \|Y_s^{(1)}\| : s > 0). \]

The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
Definition of the limit process

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\left( \frac{\log t}{t} \frac{\alpha}{\alpha-d} Z_{st}, \frac{\log t}{t} \frac{d}{\alpha-d} \frac{\log U(st)}{st} : s > 0 \right)
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\]

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Definition of the limit process

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\left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha-d}} Z_{st}, \left( \frac{\log t}{t} \right) \frac{d}{\alpha-d} \frac{\log U(st)}{st} : s > 0 \right)
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\[
\Rightarrow (Y_{s}^{(1)}, Y_{s}^{(2)} + \frac{d}{\alpha-d} (1 - \frac{1}{s}) \| Y_{s}^{(1)} \| : s > 0).
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The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
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\[
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\[
\left( \left( \frac{\log t}{t} \right)^{ \frac{\alpha}{\alpha-d} } Z_{st}, \left( \frac{\log t}{t} \right)^{ \frac{d}{\alpha-d} } \frac{\log U(st)}{st} : s > 0 \right) \Rightarrow (Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} \left(1 - \frac{1}{s}\right) \| Y_s^{(1)} \| : s > 0).
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The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
Definition of the limit process

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\left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha-d}} Z_{st}, \left( \frac{\log t}{t} \right)^{\frac{d}{\alpha-d}} \frac{\log U(st)}{st} : s > 0 \right)
\Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0 \right).
\]

The second component corresponds to the second component of the tip of the cone that defines $Y_s$. 
Definition of the limit process

\[ \left( \left( \frac{\log t}{t} \right)^\alpha, \frac{\log t}{st} \right) Z_{st}, \left( \frac{\log t}{t} \right)^d \frac{\log U(st)}{st} : s > 0 \right) \]

\[ \Rightarrow \left( Y^{(1)}_s, Y^{(2)}_s + \frac{d}{\alpha - d} \left( 1 - \frac{1}{s} \right) \| Y^{(1)}_s \| : s > 0 \right). \]

The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
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\[ \Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha - d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0 \right). \]

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Definition of the limit process

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\left(\left(\frac{\log t}{t}\right)^{\frac{\alpha}{\alpha-d}} Z_{st}, \left(\frac{\log t}{t}\right)^{\frac{d}{\alpha-d}} \frac{\log \mathcal{U}(st)}{st} : s > 0\right) \Rightarrow \left(Y_{s}^{(1)}, Y_{s}^{(2)} + \frac{d}{\alpha-d} \left(1 - \frac{1}{s}\right)\|Y_{s}^{(1)}\| : s > 0\right).
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The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
Definition of the limit process

\[
\left( \left( \frac{\log t}{t} \right)^{\alpha/d} Z_{st}, \left( \frac{\log t}{t} \right)^{\alpha/d} \frac{d}{\alpha} \frac{\log U(st)}{st} : s > 0 \right)
\]

\[\Rightarrow (Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} \left(1 - \frac{1}{s}\right) \|Y_s^{(1)}\| : s > 0).\]

The second component corresponds to the second component of the tip of the cone that defines \(Y_s\).
Definition of the limit process

\[
\left( \left( \frac{\log t}{t^d} \right)^{\frac{\alpha}{\alpha-d}} Z_{st}, \left( \frac{\log t}{t^d} \right)^{\frac{d}{\alpha-d}} \frac{\log U(st)}{st} \right) : s > 0
\]

\[
\Rightarrow (Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} (1 - \frac{1}{s}) \| Y_s^{(1)} \| : s > 0).
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Definition of the limit process

$$\left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha - d}} Z_{st}, \left( \frac{\log t}{t} \right)^{\frac{d}{\alpha - d}} \frac{\log U(st)}{st} : s > 0 \right)$$

$$\Rightarrow (Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha - d} \left(1 - \frac{1}{s}\right) \| Y_s^{(1)} \| : s > 0).$$

The second component corresponds to the second component of the tip of the cone that defines $Y_s$. 

\[ y \]
\[ x \]
Definition of the limit process

\[ \left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha-d}} Z_{st}, \left( \frac{\log t}{t} \right)^{\frac{d}{\alpha-d}} \frac{\log U(st)}{st} : s > 0 \right) \]

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\Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha - d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| \right) : s > 0.
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The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
Definition of the limit process

\[
\left( \left( \frac{\log t}{t} \right)^{\alpha/d} Z_{st}, \left( \frac{\log t}{t} \right)^{d/(\alpha-d)} \frac{\log U(st)}{st} \right) : s > 0
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\[\Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0 \right).\]

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\[\Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha - d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0 \right).\]

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The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
Definition of the limit process

\[
\left( \left( \frac{\log t}{t} \right)^{\frac{\alpha}{\alpha-d}} Z_{st}, \left( \frac{\log t}{t} \right)^{\frac{d}{\alpha-d}} \frac{\log U(st)}{st} : s > 0 \right) \\
\Rightarrow \left( Y_s^{(1)}, Y_s^{(2)} + \frac{d}{\alpha-d} \left( 1 - \frac{1}{s} \right) \| Y_s^{(1)} \| : s > 0 \right).
\]

The second component corresponds to the second component of the tip of the cone that defines \( Y_s \).
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\[
-x - \frac{d}{\alpha-d} |x|
\]
Almost sure behaviour

Recall our first theorem:

There exists a stochastic process \((Z_t : t > 0)\) with values in \(\mathbb{Z}^d\) such that

\[
\lim_{t \to \infty} \frac{u(t, Z_t)}{\sum_{z \in \mathbb{Z}^d} u(t, z)} = 1 \quad \text{in probability.}
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Question:

- How many sites are needed to support the bulk of the solution almost surely?
Two cities theorem

Theorem 3 (König, Lacoin, M, Sidorova 2007)

There exist two stochastic processes \((Z_t^{(1)} : t > 0)\) and \((Z_t^{(2)} : t > 0)\) with values in \(\mathbb{Z}^d\) such that \(Z_t^{(1)} \neq Z_t^{(2)}\) for all \(t > 0\) and

\[
\lim_{t \to \infty} \frac{u(t, Z_t^{(1)}) + u(t, Z_t^{(2)})}{\sum_{z \in \mathbb{Z}^d} u(t, z)} = 1 \quad \text{almost surely.}
\]

Remarks:
At a typical large time the mass, which is thought of as a population, inhabits one site, interpreted as a city. At some rare times, however, word spreads that a better site has been found, and the entire population moves to the new city, so that at the transition times part of the population still lives in the old city, while part has already moved to the new one.

The term two cities theorem was suggested to us by S.A. Molchanov.
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Two cities theorem: Key idea

The two cities theorem is considerably harder to prove than complete localisation, as the variational problem $\Psi_t$ does not provide a good approximation at all times.
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For a finer approximation we look at random walks which wander to a site $z$ during the time interval $[0, \rho t]$ and stay there throughout $[\rho t, t]$. This has probability

$$\approx \exp \left\{ -\|z\| \log \frac{\|z\|}{e\rho t} - 2dt + \eta(z) \right\},$$

where $\eta(z) = \log \#\{ \text{paths of length } \|z\| \text{ from origin to } z \}$. 

Peter Mörters (Bath)
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where $\eta(z) = \log \#\{ \text{paths of length } \|z\| \text{ from origin to } z \}$. We obtain

$$
\frac{1}{t} \log U(t) \approx \sup_{z \in \mathbb{Z}^d} \sup_{\rho \in (0,1)} \left\{ (1 - \rho)\xi(z) - \frac{\|z\|}{t} \log \frac{\|z\|}{e\rho t} + \frac{\eta(z)}{t} \right\}
$$

$$
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$$

$$
=: \Phi_t(z)
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Roughly speaking, if a system exhibits ageing, the probability that there is no essential change of state between time $t$ and time $t + s(t)$ is of constant order for a period $s(t)$ which depends increasingly, and often linearly, on the time $t$. Therefore, ageing can be associated to the existence of infinitely many time-scales that are inherently relevant to the system.
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Hence, as time goes on, in an ageing system changes become less likely and the typical time scales of the system are increasing. Therefore, ageing can be associated to the existence of infinitely many time-scales that are inherently relevant to the system.
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Questions:

- Does the parabolic Anderson model exhibit ageing?
- How many time-scales are relevant to our model?
Theorem 4 (M, Ortgiese, Sidorova 2009)

Let

$$v(t, x) = u(t, x) \sum_{z \in \mathbb{Z}^d} u(t, z)$$

for $t > 0, x \in \mathbb{Z}^d$.

Then there exists some $0 < \theta(c) < 1$ such that, for all $\epsilon > 0$,

$$\lim_{t \to \infty} P\{\sup_{x \in \mathbb{Z}^d} |v(t, x) - v(t, x)| < \epsilon\} = \lim_{t \to \infty} P\{\sup_{0 \leq s, x \in \mathbb{Z}^d} |v(t+s, x) - v(t, x)| < \epsilon\} = \theta(c).$$

Remark: The limit $\theta(c)$ is not associated to a generalized arc-sine law, as typically observed in simple trap models, but a more complicated function.
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$$\lim_{t \to \infty} \mathbb{P}\left\{ \sup_{x \in \mathbb{Z}^d} \left| v(t + ct, x) - v(t, x) \right| < \epsilon \right\} = \lim_{t \to \infty} \mathbb{P}\left\{ \sup_{0 \leq s \leq ct} \sup_{x \in \mathbb{Z}^d} \left| v(t + s, x) - v(t, x) \right| < \epsilon \right\} = \theta(c).$$
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Ageing: Key idea

The probability of no significant change of state between time \( t \) and \( t + ct \) can be approximated by

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Z_t = Z_{t+ct}.
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If $Z_{rt} = x$ and $\psi_t(Z_t) = y$, then this means approximately that $\Pi_t$

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If $Z_t \equiv x$ and $\frac{\psi_t(Z_t)}{a_t} = y$, then this means approximately that $\Pi_t$

- has a point in $(x, y)$ but no points $(\bar{x}, \bar{y})$ with $\bar{y} > y$,
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![Diagram](image-url)
Summary

We have seen that for a potential with heavy tails the parabolic Anderson model shows interesting extreme behaviour, in particular

- the growth rate of the total mass is asymptotically random,
- the solution is asymptotically concentrated in a single point at most times,
- this point goes to infinity at superlinear speed,
- the solution is asymptotically concentrated in two points at all times,
- the system exhibits ageing behaviour.

In the proofs we combine a very fine analysis of the random walk paths contributing in the Feynman-Kac formula with extreme value theory for the random field.

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