1. (a) Consider the following system of linear equations:

$$X + 2Y - 3Z = 1$$

$$2X + Y + Z = 2$$

$$-2X + 2Z = 3.$$

Rewrite this system in the matrix-vector form $A\,(X, Y, Z)^T = b$, and decide whether matrix A is invertible. [5]

Solution.

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The direct computation gives $\det A = -16$, hence A is invertible.

(b) Solve the system of linear equations in (a) by

(i) inverting its matrix A; [5]

Solution. First compute adjugate matrix:

$$A^* = \begin{pmatrix} 2 & -4 & 5 \\ -6 & -4 & -7 \\ 2 & -4 & -3 \end{pmatrix}.$$

Dividing elements of A^* by $\det A = -16$ we obtain the inverse matrix:

$$A^{-1} = \begin{pmatrix} -2/16 & 4/16 & -5/16 \\ 6/16 & 4/16 & 7/16 \\ -2/16 & 4/16 & 3/16 \end{pmatrix}.$$

Finally, multiplying A^{-1} by $(1, 2, 3)^T$, we obtain the solution vector $(-9/16, 35/16, 15/16)$.

(ii) using Cramer's rule; [5]

Solution. By direct computation, we get $\det A_X = 9$, $\det A_Y = -35$, and $\det A_Z = -15$. Dividing by $\det A = -16$, we obtain solutions: $X = -9/16$, $Y = 35/16$, $Z = 15/16$.

(iii) using Gaussian elimination. [5]

Solution. Taking the extended matrix to row echelon form:

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 1 & 1 & 2 \\ -2 & 0 & 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 4 & -4 & 5 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 3 & 7 & 0 \\ 0 & -4 & -4 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & -3 & 7 & 0 \\ 0 & 0 & 16/3 & 5 \end{pmatrix}$$

From the last row we get $Z = 15/16$. Using this, from the second row we get $Y = 35/16$. Finally, knowing Y and Z, from the first row we get $X = -9/16$.

2. (a) Each of the following questions asks whether or not there exists a pair of straight lines with a certain property. If so, give an example, defining each line by a system of two linear equations. If not, prove that no such lines exist.

(i) Do there exist two *distinct* straight lines ℓ and m in $\mathbb{R}^3$ for which there is no plane in $\mathbb{R}^3$ that contains both?

(ii) Do there exist two *distinct* straight lines ℓ and m in $\mathbb{R}^3$ for which there is exactly one plane in $\mathbb{R}^3$ that contains both?

(iii) Do there exist two *distinct* straight lines ℓ and m in $\mathbb{R}^3$ for which there are infinitely many *distinct* planes in $\mathbb{R}^3$ that contain both?

[3]

Solution.

(i) Yes. Example: ℓ is defined by $y = 0,\ z = 0$ while m is defined by $x = 0,\ z = 1$. There is a unique plane containing ℓ and the point $x = 0,\ y = 0,\ z = 1$ on m. This plane does not contain the whole m.

(ii) Yes. Example: ℓ is defined by $y = 0,\ z = 0$ while m is defined by $x = 0,\ z = 0$. The plane P containing ℓ and m is defined by the equation $z = 0$. Any plane Q containing ℓ and m and different from P, would intersect P by a single straight line, moreover $P \cap Q = \ell$ and $P \cap Q = m$, which is a contradiction.

(iii) No. The proof in (ii) demonstrates that there can't be even two such planes.

(b) Consider the following system of linear equations.

$$2x_1 + x_2 - 2x_3 + 3x_4 = 1$$

$$-x_1 - x_2 + 2x_3 - 3x_4 = 2$$

$$x_3 - x_4 = 3.$$

(i) Using Gaussian elimination, decide whether the system has at least one solution. If it does, represent the general solution as an affine map from one vector space to another, in matrix/vector form. Using the map find one specific solution of the system. Show your working. [5]

Solution. *Reduce the matrix of the system of equations to row echelon form:*

$$\begin{pmatrix} 2 & 1 & -2 & 3 & 1 \\ -1 & -1 & 2 & -3 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & -2 & 3 & 1 \\ 0 & -1/2 & 1 & -3/2 & 5/2 \\ 0 & 0 & 1 & -1 & 3 \end{pmatrix}.$$

Since there are no rows consisting of all zeroes except the rightmost non-zero element, the system of equations does have a solution. The original system of equations has the same set of solutions as

$$\begin{aligned} 2x_1 + x_2 - 2x_3 + 3x_4 &= 1 \\ -\frac{1}{2}x_2 + x_3 - \frac{3}{2}x_4 &= \frac{5}{2} \\ x_3 - x_4 &= 3. \end{aligned}$$

In the last equation, expressing x_4 via x_3, we get $x_4 = x_3 - 3$. Substituting the latter expression into the second equation, we get $x_3 = 4 - x_2$. Finally, from the first equation, $x_1 = 3$. Collecting these results together, we get:

$$x_1 = 3,\ x_3 = 4 - x_2,\ x_4 = 1 - x_2.$$

It follows that the general solution can be represented as the following affine map:

$$\begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}.$$

(ii) Is the affine map that you have constructed surjective? Is it injective? Justify your answers. [5]

Solution. *The map has the domain $\mathbb{R}^1$ (equipped with coordinate x_2) and the range $\mathbb{R}^3$ (equipped with coordinates $x_1,\ x_3,\ x_4$). It is not surjective since there are vectors (points) in $\mathbb{R}^3$ without pre-images in $\mathbb{R}^1$, for example $(0,0,0)$. This map is injective since values of x_3 are different for different values of x_2.*

(c) Find all eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Solution. The characteristic polynomial of A is

$$\det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 9 = (\lambda+2)(\lambda-4).$$

It follows that there are two eigenvalues, $\lambda = -2$ and $\lambda = 4$.

Consider $\lambda = -2$. Then eigenvectors (x, y) are non-zero solutions of

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., vectors of the kind $(a, -a)$ for all non-zero real numbers a.

Similar consideration of $\lambda = 4$ gives us eigenvectors of the kind (a, a) for all non-zero real numbers a. [4]

(d) Give an example of a (2×2)-matrix having no eigenvalues among real numbers.

Solution. One has to construct a matrix such that its characteristic polynomial has no real zeroes. For example,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is such a matrix because its characteristic polynomial is $\lambda^2 + 1$. [3]

3. (a) (i) Describe the set $A \subset \mathbb{R}$ of all points at which the function $f(x) = |x-1|$ is continuous. [3]

Solution Function $f(x)$ is continuous at every point $x \in \mathbb{R}$, thus $A = \mathbb{R}$.

(ii) Describe the set $B \subset \mathbb{R}$ of all points at which the function $f(x) = |x-1|$ is differentiable. [3]

Solution Function $f(x)$ is not differentiable only at $x = 1$, thus $B = \mathbb{R} \setminus \{1\}$.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at $x_0 = 0$ but not doubly differentiable at $x_0 = 0$. [3]

Solution.

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x \leq 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}.$$

(c) (i) Describe the necessary and sufficient condition on k under which the geometric series

$$\sum_{n=0}^{\infty} k^n,$$

where $k > 0$ and $k \neq 1$, converges. Prove that this condition is necessary and sufficient. [4]

Solution. The series converges if $k < 1$ and diverges otherwise. Proof is bookwork.

(ii) Describe (without proof) the necessary and sufficient condition on k under which the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k},$$

where $k > 0$, converges. [3]

Solution. Bookwork: the series converges if $k > 1$ and diverges otherwise.

(iii) Write down the first four terms of the Taylor series for $\log_e(x)$ at $x_0 = 1$. [4]

Solution.

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$