Formula and Shape

Nicolai Vorobjov

University of Bath

Bath, March 2008

Arithmetic and Geometry

Formula and Shape

Algebraic Manipulations and Geometric Imagination

Arithmetic and Geometry

Formula and Shape

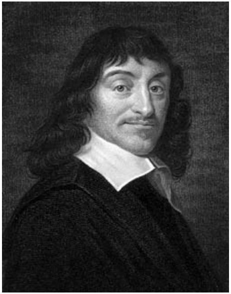
Algebraic Manipulations and Geometric Imagination

Arithmetic and Geometry

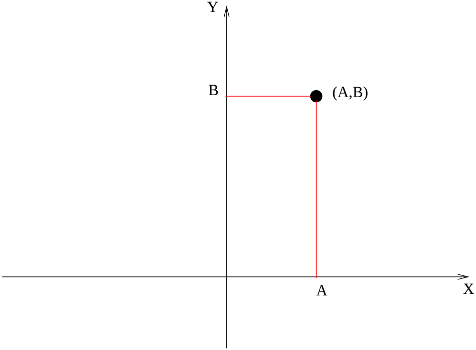
Formula and Shape

Algebraic Manipulations and Geometric Imagination

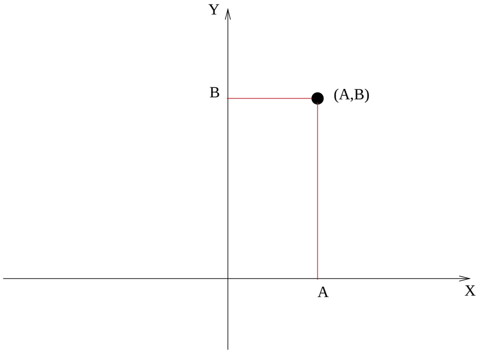
Idea of coordinates:

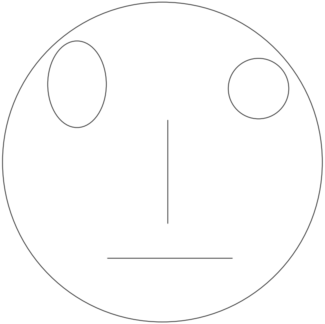
Descartes and Fermat



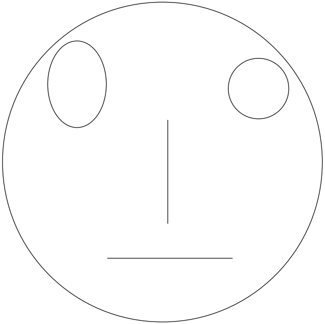
$$X - A = Y - B = 0$$



$$X - A = Y - B = 0$$

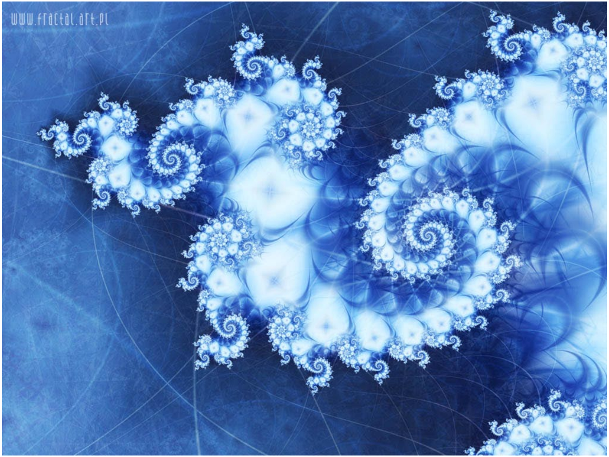


$$\left(X^2+Y^2-25\right)\left(|X-2|+|X+2|-4+(Y+3)^2\right)\left(X^2+|Y-1|+|Y+2|-3\right)$$
$$\left((X+3)^2+\frac{(Y-3/2)^2}{2}-1\right)\left((X-3)^2+(Y-2)^2-1\right)=0.$$



$$\left(X^2+Y^2-25\right)\left(|X-2|+|X+2|-4+(Y+3)^2\right)\left(X^2+|Y-1|+|Y+2|-3\right)$$
$$\left((X+3)^2+\frac{(Y-3/2)^2}{2}-1\right)((X-3)^2+(Y-2)^2-1)=0.$$

Formulae define only “tame” geometric objects, for example they can’t define a fractal:



Fundamental Principle:

A geometric object described by a “simple” formula should have a “simple shape”.

Fundamental Principle:

A geometric object described by a “simple” formula should have a “simple shape”.

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \dots, X_n$.

A polynomial in variables $X_1, \dots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$

$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \ldots, X_n$.

A polynomial in variables $X_1, \ldots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$

$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \dots, X_n$.

A polynomial in variables $X_1, \dots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$

$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \ldots, X_n$.

A polynomial in variables $X_1, \ldots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$
$$\left((X+3)^2+\frac{(Y-3/2)^2}{2}-1\right)((X-3)^2+(Y-2)^2-1)=0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \ldots, X_n$.

A polynomial in variables $X_1, \ldots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$
$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \ldots, X_n$.

A polynomial in variables $X_1, \ldots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$
$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \dots, X_n$.

A polynomial in variables $X_1, \dots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)(|X-2|+|X+2|-4+(Y+3)^2)(X^2+|Y-1|+|Y+2|-3)$$

$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \ldots, X_n$.

A polynomial in variables $X_1, \ldots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)(|X-2|+|X+2|-4+(Y+3)^2)(X^2+|Y-1|+|Y+2|-3)$$

$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2+(Y-2)^2-1) = 0.$$

Complexity of a formula

Formulae built from multivariate polynomials.

Symbols $X_1, X_2, \ldots, X_n$.

A polynomial in variables $X_1, \ldots, X_n$ is *any* expression one can construct from variables and numbers using only additions, subtractions and multiplications.

Examples:

$$X^2 + 2Y^2 - 7XY + X - 25$$

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

Not a polynomial:

$$(X^2+Y^2-25)\,(|X-2|+|X+2|-4+(Y+3)^2)\,(X^2+|Y-1|+|Y+2|-3)$$

$$\left((X+3)^2 + \frac{(Y-3/2)^2}{2} - 1\right)((X-3)^2 + (Y-2)^2 - 1) = 0.$$

Complexity measure of a polynomial?

Degree.
Every polynomial, e.g.,

$$X^3Y^5Z^{11} - 2X^2Z^8 + 3Y^7 + 100,$$

is the sum of terms with non-zero coefficients, called *monomials*.

The degree of a polynomial is the maximal number of multiplications needed to compute a monomial.

Polynomial

$$a_dX^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$$

has degree d.

Complexity measure of a polynomial?

Degree.

Every polynomial, e.g.,

$$X^3Y^5Z^{11} - 2X^2Z^8 + 3Y^7 + 100,$$

is the sum of terms with non-zero coefficients, called *monomials*.

The degree of a polynomial is the maximal number of multiplications needed to compute a monomial.

Polynomial

$$a_dX^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$$

has degree d.

Complexity measure of a polynomial?

Degree.

Every polynomial, e.g.,

$$X^3Y^5Z^{11} - 2X^2Z^8 + 3Y^7 + 100,$$

is the sum of terms with non-zero coefficients, called *monomials*.

The degree of a polynomial is the maximal number of multiplications needed to compute a monomial.

Polynomial

$$a_dX^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$$

has degree d.

Complexity measure of a polynomial?

Degree.
Every polynomial, e.g.,

$$X^3Y^5Z^{11} - 2X^2Z^8 + 3Y^7 + 100,$$

is the sum of terms with non-zero coefficients, called *monomials*.

The degree of a polynomial is the maximal number of multiplications needed to compute a monomial.

Polynomial

$$a_dX^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$$

has degree d.

Complexity measure of a polynomial?

Degree.
Every polynomial, e.g.,

$$X^3Y^5Z^{11} - 2X^2Z^8 + 3Y^7 + 100,$$

is the sum of terms with non-zero coefficients, called *monomials*.

The degree of a polynomial is the maximal number of multiplications needed to compute a monomial.

Polynomial

$$a_dX^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$$

has degree d.

Complexity measure of a polynomial?

Degree.

Every polynomial, e.g.,

$$X^3 Y^5 Z^{11} - 2X^2 Z^8 + 3Y^7 + 100,$$

is the sum of terms with non-zero coefficients, called *monomials*.

The degree of a polynomial is the maximal number of multiplications needed to compute a monomial.

Polynomial

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

has degree d.

BUT:

$$X^{100} - 1$$

is obviously a “simple” expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$X^{100} - 1$$

is obviously a “simple” expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$X^{100} - 1$$

is obviously a “simple” expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$X^{100} - 1$$

is obviously a "simple" expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$X^{100} - 1$$

is obviously a “simple” expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$X^{100} - 1$$

is obviously a "simple" expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$X^{100} - 1$$

is obviously a “simple” expression.

Number of monomials.

$$X^{100} - 1$$

has complexity 2,

$$aX^3Y^5Z^{11} - bX^2Z^8 + cY^7 + 100$$

has complexity 4 — small comparing to its degree.

A polynomial considered with this complexity measure is called *fewnomial*.

BUT:

$$(X+1)^{100}$$

is obviously a “simple” expression.
It is equal to

$$X^{100}+100X^{99}+4950X^{98}+\cdots+100X+1$$

(Newton’s binomial), so the number of monomials is large.

Additive complexity.
This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

BUT:

$$(X+1)^{100}$$

is obviously a "simple" expression.

It is equal to

$$X^{100} + 100X^{99} + 4950X^{98} + \cdots + 100X + 1$$

(Newton's binomial), so the number of monomials is large.

Additive complexity.

This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

BUT:

$$(X+1)^{100}$$

is obviously a "simple" expression.
It is equal to

$$X^{100} + 100X^{99} + 4950X^{98} + \cdots + 100X + 1$$

(Newton's binomial), so the number of monomials is large.

Additive complexity.
This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

BUT:

$$(X+1)^{100}$$

is obviously a “simple” expression.
It is equal to

$$X^{100} + 100X^{99} + 4950X^{98} + \cdots + 100X + 1$$

(Newton’s binomial), so the number of monomials is large.

Additive complexity.
This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

BUT:

$$(X+1)^{100}$$

is obviously a “simple” expression.
It is equal to

$$X^{100} + 100X^{99} + 4950X^{98} + \cdots + 100X + 1$$

(Newton’s binomial), so the number of monomials is large.

Additive complexity.
This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

BUT:

$$(X+1)^{100}$$

is obviously a “simple” expression.
It is equal to

$$X^{100} + 100X^{99} + 4950X^{98} + \cdots + 100X + 1$$

(Newton’s binomial), so the number of monomials is large.

Additive complexity.
This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

BUT:

$$(X+1)^{100}$$

is obviously a "simple" expression.
It is equal to

$$X^{100}+100X^{99}+4950X^{98}+\cdots+100X+1$$

(Newton's binomial), so the number of monomials is large.

Additive complexity.
This is the minimal number of additions or subtractions needed to *compute* the polynomial, using any number of multiplications.

Additive complexity of

$$(X+1)^{100}$$

is just 1.

When we allow also an unlimited number of divisions, then the additive complexity of the polynomial

$$X^{100} + X^{99} + X^{98} + \cdots + X + 1$$

is small since this expression is equal to

$$\frac{X^{101} - 1}{X - 1}$$

being the sum of a *geometric progression*.

When we allow also an unlimited number of divisions, then the additive complexity of the polynomial

$$X^{100} + X^{99} + X^{98} + \cdots + X + 1$$

is small since this expression is equal to

$$\frac{X^{101} - 1}{X - 1}$$

being the sum of a *geometric progression*.

When we allow also an unlimited number of divisions, then the additive complexity of the polynomial

$$X^{100} + X^{99} + X^{98} + \cdots + X + 1$$

is small since this expression is equal to

$$\frac{X^{101} - 1}{X - 1}$$

being the sum of a *geometric progression*.

Complexity of Shape

When a formula describes a finite set of points in an Euclidean space, the complexity of this set is easy to define: it's just the number of points.

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 = 0$$

Degree measure:
By the Fundamental Theorem of Algebra, the number of distinct complex numbers satisfying this equation is $\leq d$.

Then the number of real numbers satisfying the equation (number of distinct points on the straight line) is also $\leq d$.

Complexity of Shape

When a formula describes a finite set of points in an Euclidean space, the complexity of this set is easy to define: it's just the number of points.

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 = 0$$

Degree measure:

By the Fundamental Theorem of Algebra, the number of distinct complex numbers satisfying this equation is $\leq d$.

Then the number of real numbers satisfying the equation (number of distinct points on the straight line) is also $\leq d$.

Complexity of Shape

When a formula describes a finite set of points in an Euclidean space, the complexity of this set is easy to define: it's just the number of points.

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 = 0$$

Degree measure:

By the Fundamental Theorem of Algebra, the number of distinct complex numbers satisfying this equation is $\leq d$.

Then the number of real numbers satisfying the equation (number of distinct points on the straight line) is also $\leq d$.

When a formula describes a finite set of points in an Euclidean space, the complexity of this set is easy to define: it's just the number of points.

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 = 0$$

Degree measure:
By the Fundamental Theorem of Algebra, the number of distinct complex numbers satisfying this equation is $\leq d$.

Then the number of real numbers satisfying the equation (number of distinct points on the straight line) is also $\leq d$.

Complexity of Shape

When a formula describes a finite set of points in an Euclidean space, the complexity of this set is easy to define: it's just the number of points.

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 = 0$$

Degree measure:
By the Fundamental Theorem of Algebra, the number of distinct complex numbers satisfying this equation is $\leq d$.

Then the number of real numbers satisfying the equation (number of distinct points on the straight line) is also $\leq d$.

Number of monomials:

Let a univariate polynomial F have m monomials.

Descartes rule implies that the number of *positive* real solutions of the equation $F = 0$ is less than m.

Replacing X by $-X$ and adding 0, we see that the number of *all* solutions is at most $2m + 1$.

Number of monomials:
Let a univariate polynomial F have m monomials.

Descartes rule implies that the number of *positive* real solutions of the equation $F = 0$ is less than m.

Replacing X by $-X$ and adding 0, we see that the number of *all* solutions is at most $2m + 1$.

Number of monomials:

Let a univariate polynomial F have m monomials.

Descartes rule implies that the number of *positive* real solutions of the equation $F = 0$ is less than m.

Replacing X by $-X$ and adding 0, we see that the number of *all* solutions is at most $2m + 1$.

Number of monomials:
Let a univariate polynomial F have m monomials.

Descartes rule implies that the number of *positive* real solutions of the equation $F = 0$ is less than m.

Replacing X by $-X$ and adding 0, we see that the number of *all* solutions is at most $2m + 1$.

Finite set of points in the n-dimensional space, defined by a system of equations.

Degree measure:

Fundamental principle becomes Bezout Theorem:

Bezout

The number of real solutions of a generic system (conjunction) of n polynomial equations of degrees $d_1, d_2, \ldots, d_n$ respectively, in n variables, does not exceed the product $d_1 d_2 \cdots d_n$.

Finite set of points in the n-dimensional space, defined by a system of equations.

Degree measure:

Fundamental principle becomes Bezout Theorem:

Bezout

The number of real solutions of a generic system (conjunction) of n polynomial equations of degrees $d_1, d_2, \ldots, d_n$ respectively, in n variables, does not exceed the product $d_1 d_2 \cdots d_n$.

Finite set of points in the n-dimensional space, defined by a system of equations.

Degree measure:

Fundamental principle becomes Bezout Theorem:

Bezout

The number of real solutions of a generic system (conjunction) of n polynomial equations of degrees $d_1, d_2, \ldots, d_n$ respectively, in n variables, does not exceed the product $d_1 d_2 \cdots d_n$.

Finite set of points in the n-dimensional space, defined by a system of equations.

Degree measure:

Fundamental principle becomes Bezout Theorem:

Bezout

The number of real solutions of a generic *system (conjunction) of* n *polynomial equations of degrees* $d_1, d_2, \ldots, d_n$ *respectively, in* n *variables, does not exceed the product* $d_1 d_2 \cdots d_n$.

The Bezout bound is tight.

Bezout Theorem implies that *if the formula is* any *system of k polynomial equations, and maybe inequalities, and if the number of solutions is finite, then this number does not exceed*

$$d(2d-1)^{n-1}, \ \textit{where } d = \max\{d_1, \ldots, d_k\}.$$

Number of monomials:

Recent result of 1970s due to Askold Khovanskii.

The Bezout bound is tight.

Bezout Theorem implies that *if the formula is* any *system of* k *polynomial equations, and maybe inequalities, and if the number of solutions is finite, then this number does not exceed*

$$d(2d-1)^{n-1}, \text{ where } d = \max\{d_1, \dots, d_k\}.$$

Number of monomials:

Recent result of 1970s due to Askold Khovanskii.

The Bezout bound is tight.

Bezout Theorem implies that *if the formula is* any *system of* k *polynomial equations, and maybe inequalities, and if the number of solutions is finite, then this number does not exceed*

$$d(2d-1)^{n-1}, \ \textit{where } d = \max\{d_1, \dots, d_k\}.$$

Number of monomials:

Recent result of 1970s due to Askold Khovanskii.

The Bezout bound is tight.

Bezout Theorem implies that *if the formula is* any *system of* k *polynomial equations, and maybe inequalities, and if the number of solutions is finite, then this number does not exceed*

$$d(2d-1)^{n-1}, \text{ where } d = \max\{d_1, \ldots, d_k\}.$$

Number of monomials:

Recent result of 1970s due to Askold Khovanskii.

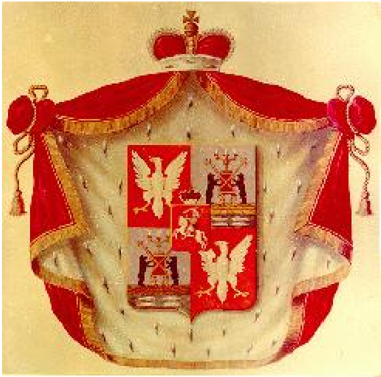
The Bezout bound is tight.

Bezout Theorem implies that *if the formula is* any *system of* k *polynomial equations, and maybe inequalities, and if the number of solutions is finite, then this number does not exceed*

$$d(2d-1)^{n-1}, \ \textit{where } d = \max\{d_1, \dots, d_k\}.$$

Number of monomials:

Recent result of 1970s due to Askold Khovanskii.



Family insignia of Prince Khovanskii

Khovanshchina (Mariinskii, St. Petersburg)

Khovanskii's Theorem

The number of real solutions with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

The number of real solutions with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

The number of real solutions with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

The number of real solutions with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.

Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

The number of real solutions with all positive coordinates *of a generic system of* n *polynomial equations in* n *variables, having* m *different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

> *The number of real solutions* with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

> *The number of real solutions* with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

> *The number of real solutions* with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

> *The number of real solutions* with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed* $2^{m(m-1)}(n+1)^m$.

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Khovanskii's Theorem

> *The number of real solutions* with all positive coordinates *of a generic system of n polynomial equations in n variables, having m different monomials in all polynomials does not exceed $2^{m(m-1)}(n+1)^m$.*

Estimate does not depend on degrees of polynomials.
Is not tight. Recent improvements by Bihan and Sottile.

Khovanskii proved much more than this: an upper bound on the number of solutions for real analytic functions satisfying triangular systems of partial differential equations with polynomial coefficients (Pfaffian functions).

Includes iterations of exponentials, trigonometric functions in appropriate domains. Fewnomials are a very special case.

Additive complexity:

Exercise :) Introduce a new variable for every addition/subtraction operation, and reduce to Khovanskii's Theorem.

Before Khovanskii, the existence of a good upper bound in terms of the additive complexity was a famous open problem in computer science.

Upper bounds in mathematics $\Leftrightarrow$ *Lower bounds* in computer science

Additive complexity:

Exercise :) Introduce a new variable for every addition/subtraction operation, and reduce to Khovanskii's Theorem.

Before Khovanskii, the existence of a good upper bound in terms of the additive complexity was a famous open problem in computer science.

Upper bounds in mathematics $\Leftrightarrow$ *Lower bounds* in computer science

Additive complexity:

Exercise :) Introduce a new variable for every addition/subtraction operation, and reduce to Khovanskii's Theorem.

Before Khovanskii, the existence of a good upper bound in terms of the additive complexity was a famous open problem in computer science.

Upper bounds in mathematics $\Leftrightarrow$ *Lower bounds* in computer science

Additive complexity:

Exercise :) Introduce a new variable for every addition/subtraction operation, and reduce to Khovanskii's Theorem.

Before Khovanskii, the existence of a good upper bound in terms of the additive complexity was a famous open problem in computer science.

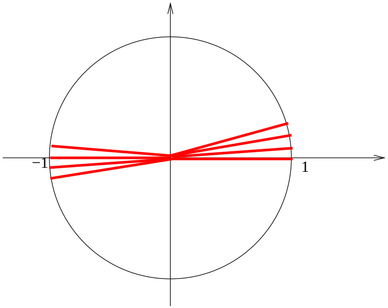
Upper bounds in mathematics $\Leftrightarrow$ *Lower bounds* in computer science

Over Complex Numbers

The *fundamental principle* does not quite work for equations over *complex numbers*, for example

$$X^{100} - 1 = 0$$

has exactly 100 different complex numbers as solutions:



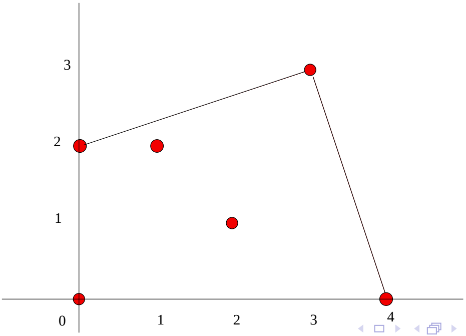
Another measure of the complexity of a polynomial: the volume of its Newton polyhedron.

For example,

$$X^3Y^3 + 2X^2Y - XY^2 + 5X^4 - 3Y^2 + 1$$

For each monomial X^iY^j draw a point with coordinates (i, j) in the plane.

Convex hull of all points is called Newton polyhedron.



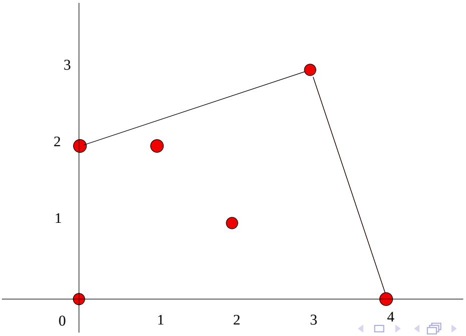
Another measure of the complexity of a polynomial: the volume of its Newton polyhedron.

For example,

$$X^3Y^3 + 2X^2Y - XY^2 + 5X^4 - 3Y^2 + 1$$

For each monomial X^iY^j draw a point with coordinates (i, j) in the plane.

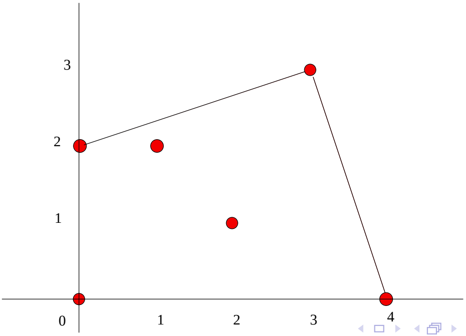
Convex hull of all points is called Newton polyhedron.



Another measure of the complexity of a polynomial: the volume of its Newton polyhedron.
For example,

$$X^3Y^3 + 2X^2Y - XY^2 + 5X^4 - 3Y^2 + 1$$

For each monomial X^iY^j draw a point with coordinates (i, j) in the plane.
Convex hull of all points is called Newton polyhedron.



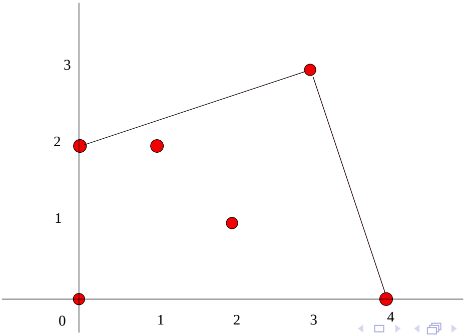
Another measure of the complexity of a polynomial: the volume of its Newton polyhedron.

For example,

$$X^3Y^3 + 2X^2Y - XY^2 + 5X^4 - 3Y^2 + 1$$

For each monomial X^iY^j draw a point with coordinates (i, j) in the plane.

Convex hull of all points is called Newton polyhedron.

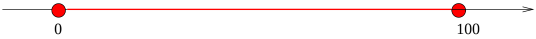


Kushnirenko's theorem:

The number of complex $\neq 0$ solutions of a generic system of n polynomial equations in n variables, having the same Newton polyhedron, does not exceed the volume of this polyhedron multiplied by $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Example 1. $X^{100} - 1 = 0$. Newton polyhedron is a segment of a straight line, and its volume is its length, and $n = 1$.

Same result as in Fundamental Theorem of Algebra.

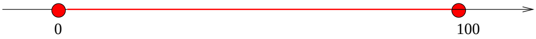


Kushnirenko's theorem:

The number of complex $\neq 0$ solutions of a generic system of n polynomial equations in n variables, having the same Newton polyhedron, does not exceed the volume of this polyhedron multiplied by $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Example 1. $X^{100} - 1 = 0$. Newton polyhedron is a segment of a straight line, and its volume is its length, and $n = 1$.

Same result as in Fundamental Theorem of Algebra.

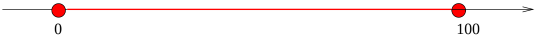


Kushnirenko's theorem:

The number of complex $\neq 0$ solutions of a generic system of n polynomial equations in n variables, having the same Newton polyhedron, does not exceed the volume of this polyhedron multiplied by $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Example 1. $X^{100} - 1 = 0$. Newton polyhedron is a segment of a straight line, and its volume is its length, and $n = 1$.

Same result as in Fundamental Theorem of Algebra.

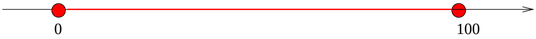


Kushnirenko's theorem:

The number of complex $\neq 0$ solutions of a generic system of n polynomial equations in n variables, having the same Newton polyhedron, does not exceed the volume of this polyhedron multiplied by $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Example 1. $X^{100} - 1 = 0$. Newton polyhedron is a segment of a straight line, and its volume is its length, and $n = 1$.

Same result as in Fundamental Theorem of Algebra.

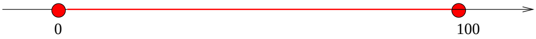


Kushnirenko's theorem:

The number of complex $\neq 0$ solutions of a generic system of n polynomial equations in n variables, having the same Newton polyhedron, does not exceed the volume of this polyhedron multiplied by $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Example 1. $X^{100} - 1 = 0$. Newton polyhedron is a segment of a straight line, and its volume is its length, and $n = 1$.

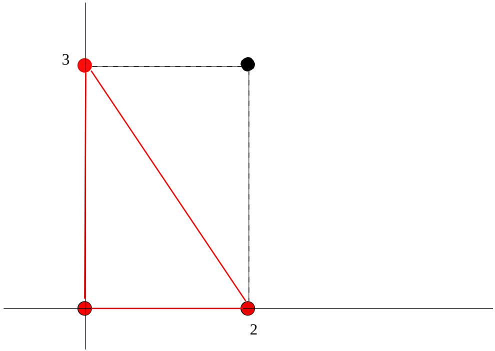
Same result as in Fundamental Theorem of Algebra.



Example 2. $X^2 + Y^3 - 1 = X^2 - Y^3 + 2 = 0$

has six complex solutions.

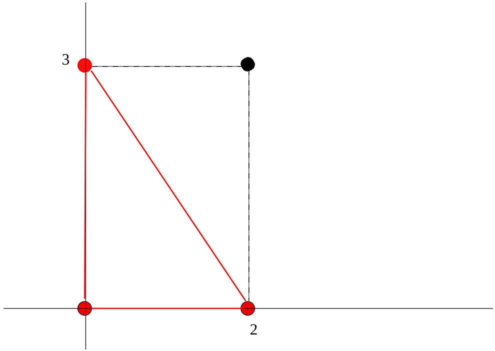
Exercise: prove it, and find them all!
(Hint: introduce new unknowns $U = X^2$ and $V = Y^3$.)



Area of Newton polygon is 3. Hence, number of solutions is 6.

Example 2. $X^2 + Y^3 - 1 = X^2 - Y^3 + 2 = 0$
has six complex solutions.

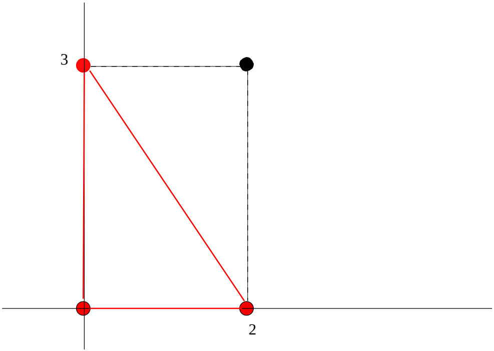
Exercise: prove it, and find them all!
(Hint: introduce new unknowns $U = X^2$ and $V = Y^3$.)



Area of Newton polygon is 3. Hence, number of solutions is 6.

Example 2. $X^2 + Y^3 - 1 = X^2 - Y^3 + 2 = 0$
has six complex solutions.

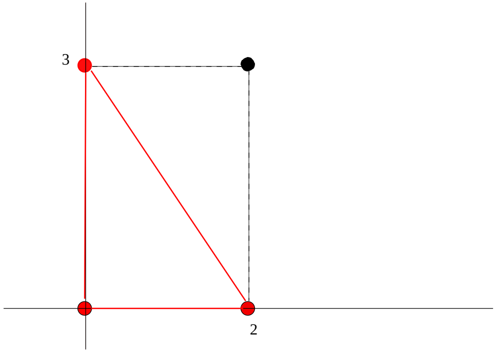
Exercise: prove it, and find them all!
(Hint: introduce new unknowns $U = X^2$ and $V = Y^3$.)



Area of Newton polygon is 3. Hence, number of solutions is 6.

Example 2. $X^2 + Y^3 - 1 = X^2 - Y^3 + 2 = 0$
has six complex solutions.

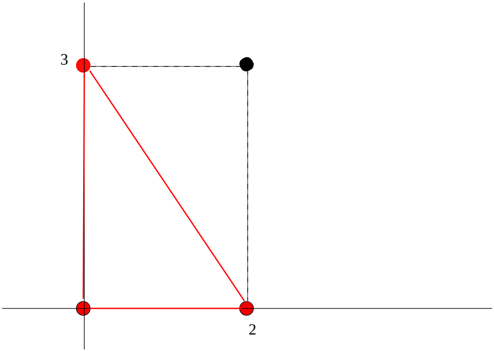
Exercise: prove it, and find them all!
(Hint: introduce new unknowns $U = X^2$ and $V = Y^3$.)



Area of Newton polygon is 3. Hence, number of solutions is 6.

Example 2. $X^2 + Y^3 - 1 = X^2 - Y^3 + 2 = 0$
has six complex solutions.

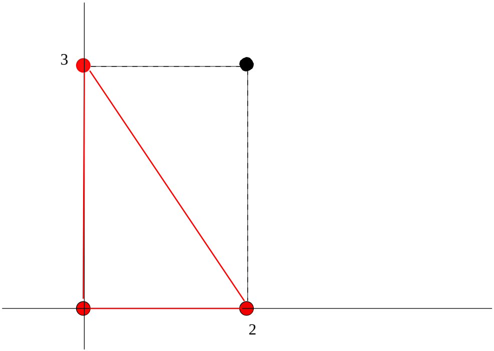
Exercise: prove it, and find them all!
(Hint: introduce new unknowns $U = X^2$ and $V = Y^3$.)



Area of Newton polygon is 3. Hence, number of solutions is 6.

Example 2. $X^2 + Y^3 - 1 = X^2 - Y^3 + 2 = 0$
has six complex solutions.

Exercise: prove it, and find them all!
(Hint: introduce new unknowns $U = X^2$ and $V = Y^3$.)



Area of Newton polygon is 3. Hence, number of solutions is 6.

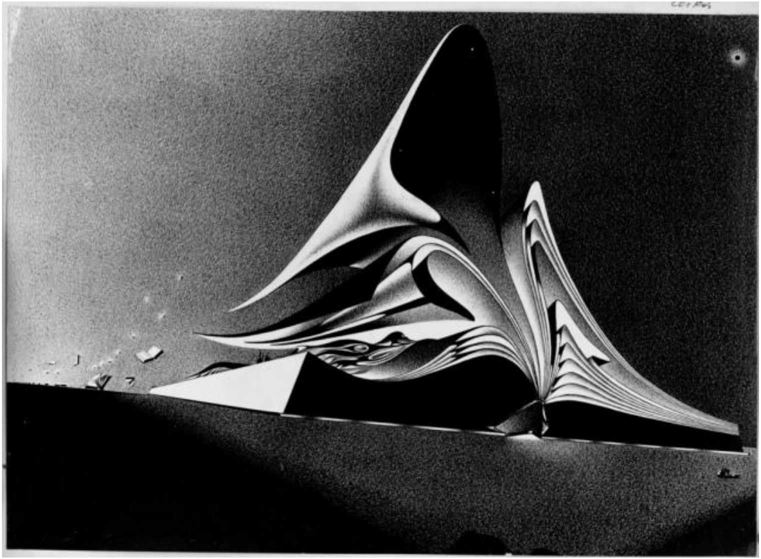
If polynomials have different Newton polyhedra, then in Kushnirenko's Theorem one should take their *mixed volume*.

Khovanskii found a common generalization of his theorem on fewnomials and Kushnirenko's Theorem.

If polynomials have different Newton polyhedra, then in Kushnirenko's Theorem one should take their *mixed volume*.

Khovanskii found a common generalization of his theorem on fewnomials and Kushnirenko's Theorem.

Higher Dimensions

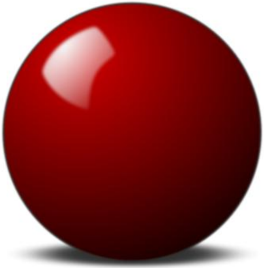


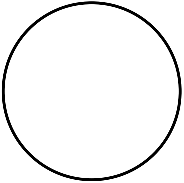
by Anatolii Fomenko

Homotopy Equivalence

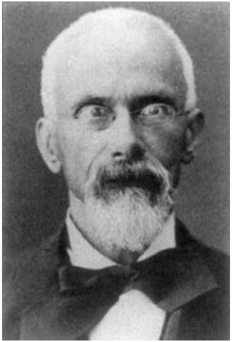
Homotopy Equivalence

Homotopy Equivalence



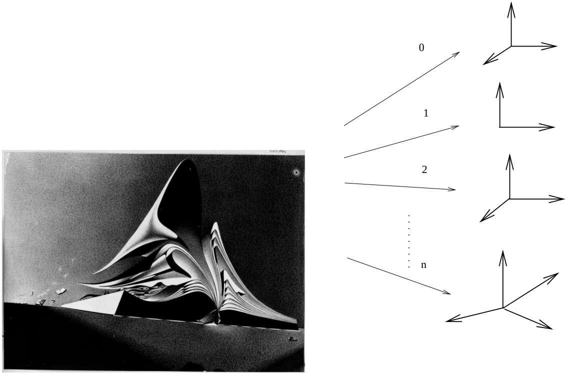


Complexity is called *the sum of Betti numbers.*



Poincaré and Betti

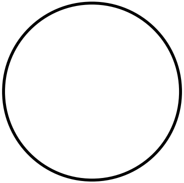
Homology Groups



The dimension of n^{th} vector space H_n is called n^{th} Betti number.

The sequence of Betti numbers is the same for homotopy equivalent spaces, for example, it is
$\dim H_0 = 1,\ \dim H_1 = 1,\ \dim H_2 = 0$ for

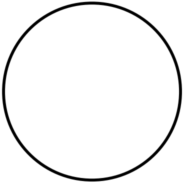
and for



The dimension of n^{th} vector space H_n is called n^{th} Betti number.

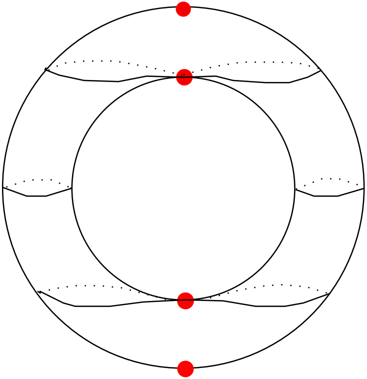
The sequence of Betti numbers is the same for homotopy equivalent spaces, for example, it is
$\dim H_0 = 1,\ \dim H_1 = 1,\ \dim H_2 = 0$ for

and for

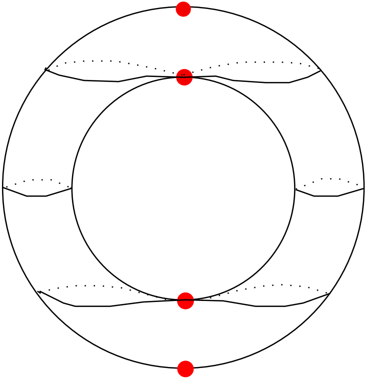


Morse Theory

Marston Morse



Rotate the surface slightly, so that the number of critical points is finite, they all lie on different levels, and moreover the surface has a non-zero curvature at each of these points.



Rotate the surface slightly, so that the number of critical points is finite, they all lie on different levels, and moreover the surface has a non-zero curvature at each of these points.

According to Morse Theory, the sum of Betti numbers does not exceed the number of critical points.

The number of critical points can be considered as a complexity measure of the surface.

The set of critical points coincides with the set of all solutions of a system of equations.
(If the surface in the example is defined by the equation $F = 0$, then this system is

$$F = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} = 0.)$$

If the set consists of points satisfying a polynomial equation of degree d in n variables, then the sum of Betti numbers (i.e., the complexity) of this set is at most

$$d(d-1)^{n-1}.$$

According to Morse Theory, the sum of Betti numbers does not exceed the number of critical points.
The number of critical points can be considered as a complexity measure of the surface.

The set of critical points coincides with the set of all solutions of a system of equations.
(If the surface in the example is defined by the equation $F = 0$, then this system is

$$F = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} = 0.)$$

If the set consists of points satisfying a polynomial equation of degree d in n variables, then the sum of Betti numbers (i.e., the complexity) of this set is at most

$$d(d-1)^{n-1}.$$

According to Morse Theory, the sum of Betti numbers does not exceed the number of critical points.
The number of critical points can be considered as a complexity measure of the surface.

The set of critical points coincides with the set of all solutions of a system of equations.
(If the surface in the example is defined by the equation $F = 0$, then this system is

$$F = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} = 0.)$$

If the set consists of points satisfying a polynomial equation of degree d in n variables, then the sum of Betti numbers (i.e., the complexity) of this set is at most

$$d(d-1)^{n-1}.$$

According to Morse Theory, the sum of Betti numbers does not exceed the number of critical points.
The number of critical points can be considered as a complexity measure of the surface.

The set of critical points coincides with the set of all solutions of a system of equations.
(If the surface in the example is defined by the equation $F = 0$, then this system is

$$F = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} = 0.)$$

If the set consists of points satisfying a polynomial equation of degree d in n variables, then the sum of Betti numbers (i.e., the complexity) of this set is at most

$$d(d-1)^{n-1}.$$

According to Morse Theory, the sum of Betti numbers does not exceed the number of critical points.
The number of critical points can be considered as a complexity measure of the surface.

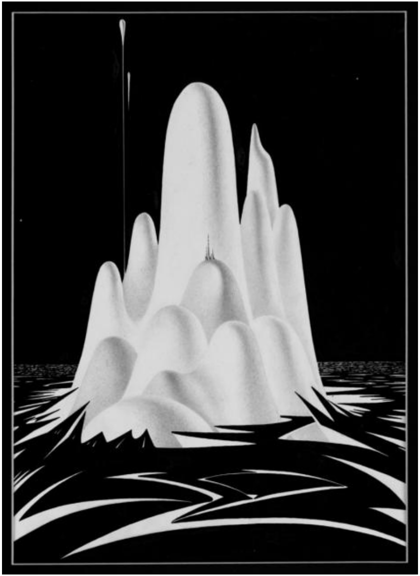
The set of critical points coincides with the set of all solutions of a system of equations.
(If the surface in the example is defined by the equation $F = 0$, then this system is

$$F = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} = 0.)$$

If the set consists of points satisfying a polynomial equation of degree d in n variables, then the sum of Betti numbers (i.e., the complexity) of this set is at most

$$d(d-1)^{n-1}.$$

Fomenko's vision of a smooth surface:



Passing to general (not necessarily smooth) sets of arbitrary dimensions defined by systems of equations is a difficult problem, a natural extension of Hilbert’s 16th problem.

Petrovskii and Oleinik

Used by Vitushkin and Kolmogorov in their solution of analytic version of Hilbert’s 13th problem.

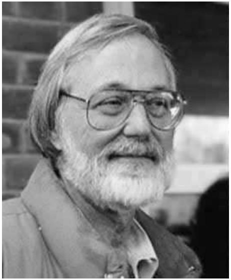
Passing to general (not necessarily smooth) sets of arbitrary dimensions defined by systems of equations is a difficult problem, a natural extension of Hilbert's 16th problem.

Petrovskii and Oleinik

Used by Vitushkin and Kolmogorov in their solution of analytic version of Hilbert's 13th problem.


1.50 FC
David Hilbert
République
Démocratique
du CONGO
2001

Hilbert



Milnor and Thom

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov

Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

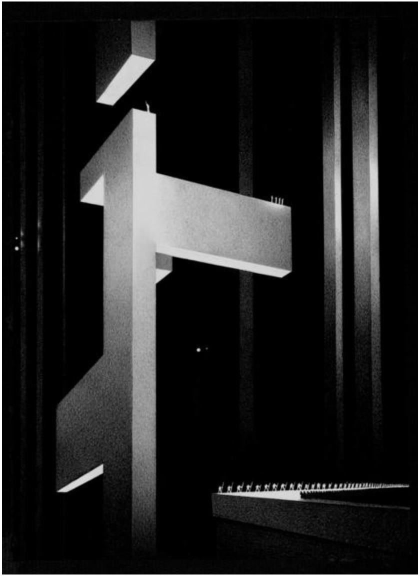
Gabrielov, Vorobjov

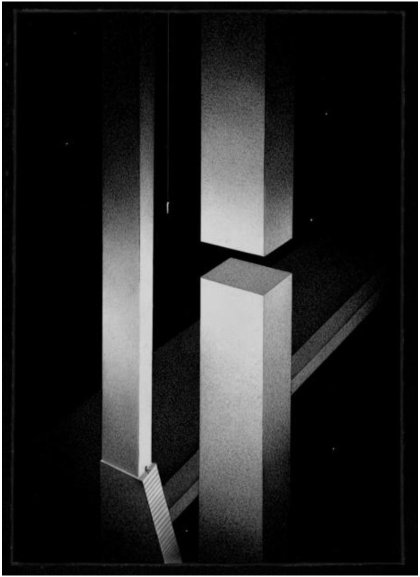
Further generalization of Petrovskii-Oleinik-Thom-Milnor bounds to sets defined by more general formulae than just conjunction of equations and inequalities.

- Unions, images under maps (projections), complements to images: first order formulae with quantifiers.
- From polynomials to more general functions, Pfaffian (including fewnomials), definable on o-minimal structures.

Basu, Pollack, Roy, Zell...

Gabrielov, Vorobjov





THE END