Fewnomials and tame topology

Nicolai Vorobjov

University of Bath

Prospects in Mathematics, Durham, December 2007

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f = a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coefficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.

  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.

- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).

  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than m $\Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by “simple” formulae should have a “simple” topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than m $\Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes' rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Ideology of the fewnomial theory

Geometric objects defined in $\mathbb{R}^n$ by “simple” formulae should have a “simple” topology.

Example

- Polynomial equation $f \equiv a_d X^d + \cdots + a_1 X + a_0 = 0$ has at most d real solutions.
  d is small $\Rightarrow$ number of the connected components of the set defined by the equation is small.
- Let the polynomial f have m monomials (terms with $\neq 0$ coeficients).
  Descartes’ rule $\Rightarrow$ The number of positive solutions of $f = 0$ is less than $m \Rightarrow$ the number of all solutions is $\leq 2m + 1$.

The theory of fewnomials is a far-reaching generalization of this example.

Complexities

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Aim: to find quantitative versions of this principle.

Two ingredients:

- complexity of a formula
- topological complexity of a geometric object.

Complexities

Geometric objects defined in $\mathbb{R}^n$ by "simple" formulae should have a "simple" topology.

Aim: to find quantitative versions of this principle.

Two ingredients:

- complexity of a formula
- topological complexity of a geometric object.

Complexities

Geometric objects defined in $\mathbb{R}^n$ by “simple” formulae should have a “simple” topology.

Aim: to find quantitative versions of this principle.

Two ingredients:

- complexity of a formula
- topological complexity of a geometric object.

Geometric objects defined in $\mathbb{R}^n$ by “simple” formulae should have a “simple” topology.

Aim: to find quantitative versions of this principle.

Two ingredients:

- complexity of a formula
- topological complexity of a geometric object.

n-variate polynomial

$$f \equiv \sum_{(i_1,\dots,i_n)} a_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n}$$

with every $a_{i_1,\dots,i_n} \in \mathbb{R}$.

What is natural to call its complexity?
n will always be a part of the complexity measure $(n, \cdot)$, what else?

Complexity of a polynomial

n-variate polynomial

$$f \equiv \sum_{(i_1,\dots,i_n)} a_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n}$$

with every $a_{i_1,\dots,i_n} \in \mathbb{R}$.

What is natural to call its complexity?

n will always be a part of the complexity measure $(n, \cdot)$, what else?

Complexity of a polynomial

n-variate polynomial

$$f \equiv \sum_{(i_1,\dots,i_n)} a_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n}$$

with every $a_{i_1,\dots,i_n} \in \mathbb{R}$.

What is natural to call its complexity?
n will always be a part of the complexity measure $(n, \cdot)$, what else?

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X - 1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X - 1) = X^{100} + \cdots + 1$

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X - 1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X - 1) = X^{100} + \cdots + 1$

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X - 1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X - 1) = X^{100} + \cdots + 1$

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X - 1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X - 1) = X^{100} + \cdots + 1$

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X - 1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X - 1) = X^{100} + \cdots + 1$

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X - 1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X - 1) = X^{100} + \cdots + 1$

Complexity of a polynomial

- Degree $d = \max_{(i_1,\dots,i_n)}(i_1 + \cdots + i_n)$.
  But $X^{100} - 1$.
- Number of monomials m. Fewnomials.
  But $(X-1)^{100}$.
- Additive complexity: a number a such that an expression representing f can be constructed using at most a additions and subtractions, and an unlimited number of multiplications (version: multiplications and divisions).
  $(X^{101} - 1)/(X-1) = X^{100} + \cdots + 1$

Bezout theorem

Polynomials $f_1, \dots, f_n \in \mathbb{C}[X_1, \dots, X_n]$ of degrees $d_1, \dots, d_n$ respectively.

A solution $\mathbf{x}$ of $f_1 = \cdots = f_n = 0$ is non-singular if

$$\det \left(\frac{\partial f_i}{\partial X_j} \right) \Bigg|_{\mathbf{x}} \neq 0$$

Theorem

The number of non-singular solutions in $\mathbb{C}^n$ of $f_1 = \cdots = f_n = 0$ is at most $d_1 \cdots d_n$.

Corollary

Same for $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$ and solutions in $\mathbb{R}^n$.

Bezout theorem

Polynomials $f_1, \dots, f_n \in \mathbb{C}[X_1, \dots, X_n]$ of degrees $d_1, \dots, d_n$ respectively.

A solution $\mathbf{x}$ of $f_1 = \cdots = f_n = 0$ is non-singular if

$$\det \left(\frac{\partial f_i}{\partial X_j} \right) \bigg|_{\mathbf{x}} \neq 0$$

Theorem

The number of non-singular solutions in $\mathbb{C}^n$ of $f_1 = \cdots = f_n = 0$ is at most $d_1 \cdots d_n$.

Corollary

Same for $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$ and solutions in $\mathbb{R}^n$.

Bezout theorem

Polynomials $f_1, \dots, f_n \in \mathbb{C}[X_1, \dots, X_n]$ of degrees $d_1, \dots, d_n$ respectively.

A solution $\mathbf{x}$ of $f_1 = \cdots = f_n = 0$ is non-singular if

$$\det \left(\frac{\partial f_i}{\partial X_j} \right) \Big|_{\mathbf{x}} \neq 0$$

Theorem

The number of non-singular solutions in $\mathbb{C}^n$ of $f_1 = \cdots = f_n = 0$ is at most $d_1 \cdots d_n$.

Corollary

Same for $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$ and solutions in $\mathbb{R}^n$.

"Bezout theorem" for fewnomials.

Polynomials $f_1, \ldots, f_n \in \mathbb{R}[X_1, \ldots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

Khovanskii's theorem for fewnomials

"Bezout theorem" for fewnomials.

Polynomials $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

"Bezout theorem" for fewnomials.

Polynomials $f_1, \ldots, f_n \in \mathbb{R}[X_1, \ldots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

Khovanskii's theorem for fewnomials

"Bezout theorem" for fewnomials.

Polynomials $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

Khovanskii's theorem for fewnomials

"Bezout theorem" for fewnomials.

Polynomials $f_1, \ldots, f_n \in \mathbb{R}[X_1, \ldots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

Khovanskii's theorem for fewnomials

"Bezout theorem" for fewnomials.

Polynomials $f_1, \ldots, f_n \in \mathbb{R}[X_1, \ldots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

Khovanskii's theorem for fewnomials

"Bezout theorem" for fewnomials.

Polynomials $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$.
Let m be the number of different monomials in all polynomials.

Theorem

The number of non-singular solutions of $f_1 = \cdots = f_n = 0$ in the positive octant of $\mathbb{R}^n$ is at most $2^{m(m-1)}(n+1)^m$.

Better bounds by Bihan and Sottile.

Exercise

Deduce an upper bound for the number of non-singular solutions in terms of the additive complexity.

Khovanskii actually proved a bound for much more general functions f_i than polynomials, Pfaffian functions.

Pfaffian functions are real analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.
Include polynomials, algebraic, elementary transcendental functions and their compositions (in appropriate domains).

Example

Exponential polynomial

$$e^{e^{f_1(X_1,\ldots,X_n)}+e^{f_2(X_1,\ldots,X_n)}} - e^{f_3(X_1,\ldots,X_n)} + f_4(X_1,\ldots,X_n),$$

with polynomials f_i, is a Pfaffian function in $\mathbb{R}^n$.
Natural complexity measure.

Pfaffian functions

Khovanskii actually proved a bound for much more general functions f_i than polynomials, Pfaffian functions.

Pfaffian functions are real analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.
Include polynomials, algebraic, elementary transcendental functions and their compositions (in appropriate domains).

Example

Exponential polynomial

$$e^{e^{f_1(X_1,\dots,X_n)}+e^{f_2(X_1,\dots,X_n)}} - e^{f_3(X_1,\dots,X_n)} + f_4(X_1,\dots,X_n),$$

with polynomials f_i, is a Pfaffian function in $\mathbb{R}^n$.
Natural complexity measure.

Pfaffian functions

Khovanskii actually proved a bound for much more general functions f_i than polynomials, Pfaffian functions.

Pfaffian functions are real analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.
Include polynomials, algebraic, elementary transcendental functions and their compositions (in appropriate domains).

Example

Exponential polynomial

$$e^{e^{f_1(X_1,\dots,X_n)}+e^{f_2(X_1,\dots,X_n)}} - e^{f_3(X_1,\dots,X_n)} + f_4(X_1,\dots,X_n),$$

with polynomials f_i, is a Pfaffian function in $\mathbb{R}^n$.
Natural complexity measure.

Khovanskii actually proved a bound for much more general functions f_i than polynomials, Pfaffian functions.

Pfaffian functions are real analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.
Include polynomials, algebraic, elementary transcendental functions and their compositions (in appropriate domains).

Example

Exponential polynomial

$$e^{e^{f_1(X_1,\dots,X_n)}+e^{f_2(X_1,\dots,X_n)}} - e^{f_3(X_1,\dots,X_n)} + f_4(X_1,\dots,X_n),$$

with polynomials f_i, is a Pfaffian function in $\mathbb{R}^n$.
Natural complexity measure.

Pfaffian functions

Khovanskii actually proved a bound for much more general functions f_i than polynomials, Pfaffian functions.

Pfaffian functions are real analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.
Include polynomials, algebraic, elementary transcendental functions and their compositions (in appropriate domains).

Example

Exponential polynomial

$$e^{e^{f_1(X_1,\dots,X_n)}+e^{f_2(X_1,\dots,X_n)}} - e^{f_3(X_1,\dots,X_n)} + f_4(X_1,\dots,X_n),$$

with polynomials f_i, is a Pfaffian function in $\mathbb{R}^n$.
Natural complexity measure.

Khovanskii actually proved a bound for much more general functions f_i than polynomials, Pfaffian functions.

Pfaffian functions are real analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.
Include polynomials, algebraic, elementary transcendental functions and their compositions (in appropriate domains).

Example

Exponential polynomial

$$e^{e^{f_1(X_1,\dots,X_n)}+e^{f_2(X_1,\dots,X_n)}} - e^{f_3(X_1,\dots,X_n)} + f_4(X_1,\dots,X_n),$$

with polynomials f_i, is a Pfaffian function in $\mathbb{R}^n$.
Natural complexity measure.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.
Coordinate change: $X_i \to e^{Y_i}$.
In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.
Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.
Coordinate change: $X_i \to e^{Y_i}$.
In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.
Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.
Coordinate change: $X_i \to e^{Y_i}$.
In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.
Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.

Coordinate change: $X_i \to e^{Y_i}$.

In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.

Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.
Coordinate change: $X_i \to e^{Y_i}$.
In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.
Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.
Coordinate change: $X_i \to e^{Y_i}$.
In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.
Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

Khovanskii's theorem for Pfaffian functions

Natural complexity measure for systems of differential equations induces the complexity on Pfaffian functions.

Khovanskii's theorem is true for systems of equations $f_1 = \cdots = f_n = 0$, where f_i are Pfaffian functions having common domain.

Example

How to prove the bound for fewnomials.
Coordinate change: $X_i \to e^{Y_i}$.
In the polynomial each monomial $X_1^{i_1} \cdots X_n^{i_n}$ will be replaced by $e^{i_1 Y_1 + \cdots + i_n Y_n}$.
Then apply Khovanskii's theorem to the resulting system of Pfaffian functions in $\mathbb{R}^n$.

First order formulae

Sets defined in $\mathbb{R}^n$ by systems of polynomial equations (i.e., intersections of sets of the kind $\{f = 0\}$ with $f \in \mathbb{R}[X_1, \ldots, X_n]$) are called real algebraic.

Sets in $\mathbb{R}^n$ defined by Boolean combinations of equations and inequalities (i.e., arbitrary unions, intersections and complements of sets of the kind $\{f = 0\}$, $\{f > 0\}$) are called semialgebraic.
Similar: semi-Pfaffian sets

First order formulae

Sets defined in $\mathbb{R}^n$ by systems of polynomial equations (i.e., intersections of sets of the kind $\{f = 0\}$ with $f \in \mathbb{R}[X_1, \ldots, X_n]$) are called real algebraic.

Sets in $\mathbb{R}^n$ defined by Boolean combinations of equations and inequalities (i.e., arbitrary unions, intersections and complements of sets of the kind $\{f = 0\}$, $\{f > 0\}$) are called semialgebraic.

Similar: semi-Pfaffian sets

First order formulae

Sets defined in $\mathbb{R}^n$ by systems of polynomial equations (i.e., intersections of sets of the kind $\{f = 0\}$ with $f \in \mathbb{R}[X_1, \dots, X_n]$) are called real algebraic.

Sets in $\mathbb{R}^n$ defined by Boolean combinations of equations and inequalities (i.e., arbitrary unions, intersections and complements of sets of the kind $\{f = 0\}$, $\{f > 0\}$) are called semialgebraic.
Similar: semi-Pfaffian sets

Tarski's theorem

Theorem

Sets in $\mathbb{R}^n$ defined by formulae of the first order theory of $\mathbb{R}$ (i.e., projections on $\mathbb{R}^n$ of semialgebraic sets in $\mathbb{R}^{n+k}$ for some $k \geq 0$) are exactly semialgebraic sets.

This is wrong for semi-Pfaffian sets!
Hence, for projections of semi-Pfaffian sets we have to use a new term, sub-Pfaffian sets.
The family of all sub-Pfaffian sets in $\mathbb{R}^n$ is closed under Boolean operations.

Complexity naturally extends from polynomials (or Pfaffian functions) to first order formulae involving these functions.

Theorem

Sets in $\mathbb{R}^n$ defined by formulae of the first order theory of $\mathbb{R}$ (i.e., projections on $\mathbb{R}^n$ of semialgebraic sets in $\mathbb{R}^{n+k}$ for some $k \geq 0$) are exactly semialgebraic sets.

This is wrong for semi-Pfaffian sets!
Hence, for projections of semi-Pfaffian sets we have to use a new term, sub-Pfaffian sets.
The family of all sub-Pfaffian sets in $\mathbb{R}^n$ is closed under Boolean operations.

Complexity naturally extends from polynomials (or Pfaffian functions) to first order formulae involving these functions.

Tarski's theorem

Theorem

Sets in $\mathbb{R}^n$ defined by formulae of the first order theory of $\mathbb{R}$ (i.e., projections on $\mathbb{R}^n$ of semialgebraic sets in $\mathbb{R}^{n+k}$ for some $k \geq 0$) are exactly semialgebraic sets.

This is wrong for semi-Pfaffian sets!

Hence, for projections of semi-Pfaffian sets we have to use a new term, sub-Pfaffian sets.

The family of all sub-Pfaffian sets in $\mathbb{R}^n$ is closed under Boolean operations.

Complexity naturally extends from polynomials (or Pfaffian functions) to first order formulae involving these functions.

Tarski's theorem

Theorem

Sets in $\mathbb{R}^n$ defined by formulae of the first order theory of $\mathbb{R}$ (i.e., projections on $\mathbb{R}^n$ of semialgebraic sets in $\mathbb{R}^{n+k}$ for some $k \geq 0$) are exactly semialgebraic sets.

This is wrong for semi-Pfaffian sets!
Hence, for projections of semi-Pfaffian sets we have to use a new term, sub-Pfaffian sets.
The family of all sub-Pfaffian sets in $\mathbb{R}^n$ is closed under Boolean operations.

Complexity naturally extends from polynomials (or Pfaffian functions) to first order formulae involving these functions.

Theorem

Sets in $\mathbb{R}^n$ defined by formulae of the first order theory of $\mathbb{R}$ (i.e., projections on $\mathbb{R}^n$ of semialgebraic sets in $\mathbb{R}^{n+k}$ for some $k \geq 0$) are exactly semialgebraic sets.

This is wrong for semi-Pfaffian sets!
Hence, for projections of semi-Pfaffian sets we have to use a new term, sub-Pfaffian sets.
The family of all sub-Pfaffian sets in $\mathbb{R}^n$ is closed under Boolean operations.

Complexity naturally extends from polynomials (or Pfaffian functions) to first order formulae involving these functions.

Theorem

Sets in $\mathbb{R}^n$ defined by formulae of the first order theory of $\mathbb{R}$ (i.e., projections on $\mathbb{R}^n$ of semialgebraic sets in $\mathbb{R}^{n+k}$ for some $k \geq 0$) are exactly semialgebraic sets.

This is wrong for semi-Pfaffian sets!
Hence, for projections of semi-Pfaffian sets we have to use a new term, sub-Pfaffian sets.
The family of all sub-Pfaffian sets in $\mathbb{R}^n$ is closed under Boolean operations.

Complexity naturally extends from polynomials (or Pfaffian functions) to first order formulae involving these functions.

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.
- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

Closure in $\mathbb{R}^n$ of definable is definable.
Connected is path-connected.

- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.

- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

Tameness of definable sets

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.

- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

Tameness of definable sets

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.
- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

Tameness of definable sets

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.

- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

Tameness of definable sets

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.
- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

Tameness of definable sets

The topology of definable sets in $\mathbb{R}^n$ (semialgebraic, semi-Pfaffian, sub-Pfaffian) is "tame".

- No "pathological" objects, like

$$\{Y = \sin(1/X)\} \cap \{X > 0\} \subset \mathbb{R}^2.$$

  Closure in $\mathbb{R}^n$ of definable is definable.
  Connected is path-connected.
- Finite triangulation.
- Finite smooth stratification.
- Finite number of topological types in a definable family.

Complexity of a definable set

A natural measure for the complexity of a topological space $S \subset \mathbb{R}^n$ is the sequence of ranks of it's homology groups

$\mathrm{b}_i(S) = \mathrm{rank}\, H_i(S, \mathbb{Q})$ (Betti numbers),

or the sum of ranks

$\mathrm{b}_*(S) = \sum_i b_i(S)$ (total Betti number).

Because of tameness, it does not matter which homology theory is used.

Complexity of a definable set

A natural measure for the complexity of a topological space $S \subset \mathbb{R}^n$ is the sequence of ranks of it's homology groups

$\mathrm{b}_i(S) = \mathrm{rank}\, H_i(S, \mathbb{Q})$ (Betti numbers),

or the sum of ranks

$\mathrm{b}_*(S) = \sum_i b_i(S)$ (total Betti number).

Because of tameness, it does not matter which homology theory is used.

Complexity of a definable set

A natural measure for the complexity of a topological space $S \subset \mathbb{R}^n$ is the sequence of ranks of it's homology groups

$\mathrm{b}_i(S) = \mathrm{rank}\, H_i(S, \mathbb{Q})$ (Betti numbers),

or the sum of ranks

$\mathrm{b}_*(S) = \sum_i b_i(S)$ (total Betti number).

Because of tameness, it does not matter which homology theory is used.

Complexity of a definable set

A natural measure for the complexity of a topological space $S \subset \mathbb{R}^n$ is the sequence of ranks of it's homology groups

$\mathrm{b}_i(S) = \mathrm{rank}\, H_i(S, \mathbb{Q})$ (Betti numbers),

or the sum of ranks

$\mathrm{b}_*(S) = \sum_i b_i(S)$ (total Betti number).

Because of tameness, it does not matter which homology theory is used.

Quantitative version of

Definable sets in $\mathbb{R}^n$ defined by "simple" formulae should have a "simple" topology.

Aim: obtain tight (enough) upper bound on the complexity of definable sets as an explicit function of the complexity of the defining formulae.

EXAMPLE: Thom–Milnor upper bound on b_* for real algebraic sets defined by polynomials or by fewnomials.

Quantitative version of

Definable sets in $\mathbb{R}^n$ defined by "simple" formulae should have a "simple" topology.

Aim: obtain tight (enough) upper bound on the complexity of definable sets as an explicit function of the complexity of the defining formulae.

EXAMPLE: Thom–Milnor upper bound on b_* for real algebraic sets defined by polynomials or by fewnomials.

Effective fewnomial theory

Quantitative version of

Definable sets in $\mathbb{R}^n$ defined by "simple" formulae should have a "simple" topology.

Aim: obtain tight (enough) upper bound on the complexity of definable sets as an explicit function of the complexity of the defining formulae.

EXAMPLE: Thom–Milnor upper bound on b_* for real algebraic sets defined by polynomials or by fewnomials.

Effective fewnomial theory

Quantitative version of

Definable sets in $\mathbb{R}^n$ defined by "simple" formulae should have a "simple" topology.

Aim: obtain tight (enough) upper bound on the complexity of definable sets as an explicit function of the complexity of the defining formulae.

EXAMPLE: Thom–Milnor upper bound on b_* for real algebraic sets defined by polynomials or by fewnomials.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \leq d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \leq d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$, *then* $\mathrm{b}_*(S) \leq 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \le d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \le d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$, *then* $\mathrm{b}_*(S) \le 2^{(n+m)(n+m-1)/2+1}(2d-1)^{n+m-1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \leq d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \leq d(2d-1)^{n-1}$.
- *If $S \subset \mathbb{R}^n_+$, then $\mathrm{b}_*(S) \leq 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.*

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \le d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \le d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$, *then* $\mathrm{b}_*(S) \le 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1)\cdots(X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \leq d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \leq d(2d-1)^{n-1}$.
- *If $S \subset \mathbb{R}^n_+$, then* $\mathrm{b}_*(S) \leq 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \leq d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \leq d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$, *then* $\mathrm{b}_*(S) \leq 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \le d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \le d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$, *then* $\mathrm{b}_*(S) \le 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \leq d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \leq d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$*, then* $\mathrm{b}_*(S) \leq 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Thom-Milnor bound

Let $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$ be an algebraic set, $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $\deg f_i \leq d$, the number of different monomials in all f_i is m (ignoring differences in $\neq 0$ coefficients!).

Theorem

- $\mathrm{b}_*(S) \leq d(2d-1)^{n-1}$.
- *If* $S \subset \mathbb{R}^n_+$, *then* $\mathrm{b}_*(S) \leq 2^{(n+m)(n+m-1)/2+1}(2n+1)^{n+m+1}$.

(Note that for $f_i = (X_i - 1) \cdots (X_i - d)$ and $k = n$, $\mathrm{b}_*(S) = d^n$).

Sketch of Milnor's proof.

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f/\partial X_1, \ldots, \partial f/\partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n. Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f/\partial X_1, \ldots, \partial f/\partial X_{n-1})(\mathbf{x}) = 0$.) A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f/\partial X_1, \dots, \partial f/\partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n. Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f/\partial X_1, \dots, \partial f/\partial X_{n-1})(\mathbf{x}) = 0$.) A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f/\partial X_1, \dots, \partial f/\partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n. Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f/\partial X_1, \dots, \partial f/\partial X_{n-1})(\mathbf{x}) = 0$.) A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f / \partial X_1, \ldots, \partial f / \partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n. Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f / \partial X_1, \ldots, \partial f / \partial X_{n-1})(\mathbf{x}) = 0$.) A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f/\partial X_1, \ldots, \partial f/\partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n.
Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f/\partial X_1, \ldots, \partial f/\partial X_{n-1})(\mathbf{x}) = 0$.)
A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f/\partial X_1, \ldots, \partial f/\partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n. Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f/\partial X_1, \ldots, \partial f/\partial X_{n-1})(\mathbf{x}) = 0$.)
A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

$S = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n$

If $k = n$ and all points in S are nonsingular, then Theorem immediately follows from Bezout and Khovanskii.

Let S be a compact nonsingular hypersurface: $S = \{f = 0\}$ with $(\partial f/\partial X_1, \ldots, \partial f/\partial X_n)(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in S$.

Morse Theory

Consider $\pi : S \to \mathbb{R}$, the projection of S on the coordinate X_n. Critical points of π: tangent points $\mathbf{x}$ on S of the sweeping hyperplane $X_n = \mathrm{const}$ (i.e., $(\partial f/\partial X_1, \ldots, \partial f/\partial X_{n-1})(\mathbf{x}) = 0$.) A critical point $\mathbf{x}$ is non-degenerate if Gaussian curvature of S at $\mathbf{x}$ is $\neq 0$ (i.e., the Hessian $\neq 0$).

Compact nonsingular hypersurface

Rotating S if needed, we can assume that all critical points of π are non-degenerate and all critical values are distinct.

According to Morse Theory, $\mathrm{b}_*(S) \leq$ number of all critical points, provided all are non-degenerate.

Set of all critical points is $\{f = \partial f / \partial X_1 = \cdots = \partial f / \partial X_{n-1} = 0\}$.

Theorem now follows from Bezout or Khovanskii respectively.

Compact nonsingular hypersurface

Rotating S if needed, we can assume that all critical points of π are non-degenerate and all critical values are distinct.

According to Morse Theory, $\mathrm{b}_*(S) \leq$ number of all critical points, provided all are non-degenerate.

Set of all critical points is $\{f = \partial f/\partial X_1 = \cdots = \partial f/\partial X_{n-1} = 0\}$.

Theorem now follows from Bezout or Khovanskii respectively.

Compact nonsingular hypersurface

Rotating S if needed, we can assume that all critical points of π are non-degenerate and all critical values are distinct.

According to Morse Theory, $\mathrm{b}_*(S) \leq$ number of all critical points, provided all are non-degenerate.

Set of all critical points is $\{f = \partial f / \partial X_1 = \cdots = \partial f / \partial X_{n-1} = 0\}$.

Theorem now follows from Bezout or Khovanskii respectively.

Compact nonsingular hypersurface

Rotating S if needed, we can assume that all critical points of π are non-degenerate and all critical values are distinct.

According to Morse Theory, $\mathrm{b}_*(S) \leq$ number of all critical points, provided all are non-degenerate.

Set of all critical points is $\{f = \partial f / \partial X_1 = \cdots = \partial f / \partial X_{n-1} = 0\}$.

Theorem now follows from Bezout or Khovanskii respectively.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.

For $R \in \mathbb{R}$, $R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.

$f = f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.

So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.

$f = f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

Arbitrary algebraic set

Let $S = \{f_1 = \cdots = f_k = 0\}$.
For $R \in \mathbb{R},\ R > 0$ large enough, $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ is homotopy equivalent to S, and is compact.
$f \equiv f_1^2 + \cdots + f_k^2 + \varepsilon(X_1^2 + \cdots + X_n^2 - R)$

Lemma

For all sufficiently small $\varepsilon > 0$ the sets $S \cap \{X_1^2 + \cdots + X_n^2 \leq R\}$ and $\{f \leq 0\}$ are homotopy equivalent.

Thus, it's sufficient to bound $\mathrm{b}_*(\{f \leq 0\})$.
So far we can bound $\mathrm{b}_*(\{f = 0\}) = \mathrm{b}_*(\{\partial\{f \leq 0\}\})$, since $\{f = 0\}$ is a compact nonsingular hypersurface (check!).

Lemma

$\mathrm{b}_*(\{f \leq 0\}) \leq \frac{1}{2}\mathrm{b}_*(\{f = 0\})$.

Thom-Milnor bound follows.

General definable sets

Sets S definable by systems of inequalities or unions of sets defined by systems of inequalities, or projections of such sets.

In general, sets in $\mathbb{R}^n$ satisfying formulae of the kind

$$\forall X^{(1)} \exists X^{(2)} \forall X^{(3)} \dots \exists X^{(s)} F(X, X^{(1)}, \dots, X^{(s)}),$$

where $X^{(i)} = (X_{i1}, \dots, X_{is_i})$, $X = (X_1, \dots, X_n)$, and F is a Boolean combination of equations and inequalities.

Some serious advances were made in "fewnomial principle" with relation to sets definable by such formulae.

General definable sets

Sets S definable by systems of inequalities or unions of sets defined by systems of inequalities, or projections of such sets.

In general, sets in $\mathbb{R}^n$ satisfying formulae of the kind

$$\forall X^{(1)} \exists X^{(2)} \forall X^{(3)} \dots \exists X^{(s)} F(X, X^{(1)}, \dots, X^{(s)}),$$

where $X^{(i)} = (X_{i1}, \dots, X_{is_i})$, $X = (X_1, \dots, X_n)$, and F is a Boolean combination of equations and inequalities.

Some serious advances were made in "fewnomial principle" with relation to sets definable by such formulae.

General definable sets

Sets S definable by systems of inequalities or unions of sets defined by systems of inequalities, or projections of such sets.

In general, sets in $\mathbb{R}^n$ satisfying formulae of the kind

$$\forall X^{(1)} \exists X^{(2)} \forall X^{(3)} \dots \exists X^{(s)} F(X, X^{(1)}, \dots, X^{(s)}),$$

where $X^{(i)} = (X_{i1}, \dots, X_{is_i}),\ X = (X_1, \dots, X_n)$, and F is a Boolean combination of equations and inequalities.

Some serious advances were made in “fewnomial principle” with relation to sets definable by such formulae.

General definable sets

Sets S definable by systems of inequalities or unions of sets defined by systems of inequalities, or projections of such sets.

In general, sets in $\mathbb{R}^n$ satisfying formulae of the kind

$$\forall X^{(1)} \exists X^{(2)} \forall X^{(3)} \dots \exists X^{(s)} F(X, X^{(1)}, \dots, X^{(s)}),$$

where $X^{(i)} = (X_{i1}, \dots, X_{is_i})$, $X = (X_1, \dots, X_n)$, and F is a Boolean combination of equations and inequalities.

Some serious advances were made in “fewnomial principle” with relation to sets definable by such formulae.

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

Prospects

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

Prospects

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

Prospects

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

Prospects

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

Prospects

Probably very hard open problems:

- Upper bounds on triangulations of definable sets.
- Upper bounds on smooth stratifications.

More accessible questions:

- Global upper bounds for fewnomials.
- Upper bounds for for sets described by definable functions with axiomatic concept of complexity.

References

1. A. Khovanskii, *Fewnomials*, AMS, 1991.

2. A. Gabrielov, N. Vorobjov, Complexity of computations with Pfaffian and Noetherian functions, Kluwer, 2004, http://www.bath.ac.uk/∼masnnv