What is . . . a coalitional game?

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Bath

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Forgotten Game Theory

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Coalitional Game

Definition

A coalitional game is system

$$\Gamma = (I, \mathcal{K}_{ac}, \{S_i\}_{i \in I}, \mathcal{K}_{in}, \{H_K\}_{K \in \mathcal{K}_{in}}),$$

where

- I is a finite set of players.
- $\mathcal{K}_{in}, \mathcal{K}_{ac}$ (with $\mathcal{K}_{in} \subset \mathcal{K}_{ac}$) are families of subsets of I.
  $K \in \mathcal{K}_{ac}$ is called *action* coalition,
  $K \in \mathcal{K}_{in}$ is called *interest* coalition.
- $\{S_i\}_{i \in I}$ is the family of finite sets of strategies.
- $\{H_K\}_{K \in \mathcal{K}_{in}}$ is a family of payoff functions

$$H_K : \prod_{i \in I} S_i \to \mathbb{R}.$$

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Coalitional strategies

Consider a coalition $K \subset I$, let $S_K := \prod_{i \in K} S_i$.

Element $f_K \in S_K$ is a coalitional strategy, a function $f_K : K \to S_K$ such that $f_K(i) \in S_i$.

Coalitional strategies $f_K \in S_K$ and $f_L \in S_L$ are compatible if $f_K = f_L$ on $K \cap L$.

If $f_K \in S_K$ and $f_L \in S_L$, then $f_K \| f_L$ is the function in S_K compatible with restriction of f_K on $K \setminus L$ and with restriction of f_L on $K \cap L$.

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Extension of coalitional strategies

Let $\mathcal{K}$ be a family of coalitions of I.

The the corresponding family $f_{\mathcal{K}}$ of coalitional strategies is called compatible if all its functions are pair-wise compatible.

The function f_I (situation) is the extension of the compatible family $f_{\mathcal{K}}$ if f_I is compatible with every function in $f_{\mathcal{K}}$.

If $I = K \cup L,\ K \cap L = \emptyset$, we write the extension as $f_I = (f_K, f_L)$.

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Optimality Principle: Admissible Situations

Consider $\Gamma = (I, \mathcal{K}_{ac}, \{S_i\}_{i \in I}, \mathcal{K}_{in}, \{H_K\}_{K \in \mathcal{K}_{in}})$.

Let $K \in \mathcal{K}_{ac}$, $\widetilde{K} \in \mathcal{K}_{in}$ with $K \cup \widetilde{K} \in \mathcal{K}_{ac}$.

Definition

A situation f_I is called admissible for $\widetilde{K}$ relative to K

$$H_{\widetilde{K}}(f_I \| f_{K \cup \widetilde{K}}) \leq H_{\widetilde{K}}(f_I \| f_K)$$

for all $f_K \in S_K$ and all $f_{\widetilde{K} \setminus K} \in S_{\widetilde{K} \setminus K}$ (where $f_{K \cup \widetilde{K}} = (f_{\widetilde{K} \setminus K}, f_K)$).

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Definition

A situation f_I is called admissible for $\widetilde{K}$ relative to K if

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Consider $\varphi : \mathcal{K}_{in} \to \mathcal{K}_{ac}$ such that $K \cup \varphi(K) \in \mathcal{K}_{ac}$ for every $K \in \mathcal{K}_{in}$.

Definition

Situation f_l is called φ-stable if f_l is admissible every $K \in \mathcal{K}_{in}$ relative to $\varphi(K)$.

Let $\varphi(K) = \emptyset$ for every $K \in \mathcal{K}_{in}$. Then φ-stable situations are called equilibrium situations.

Example

If Γ is a non-cooperative game, i.e., $\dim(\mathcal{K}_{ac}) = 0$ and $\mathcal{K}_{in} = \mathcal{K}_{ac} \setminus \{\emptyset\}$, then equilibrium situations are Nash equilibrium situations.

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Example

$$I = \{1,2,3\},\ \mathcal{K}_{ac} = \{\emptyset, 1, 2, 3, \{1,2\}, \{2,3\}\},$$

$$\mathcal{K}_{in} = \{\{1,2\}, 3\},\ \varphi(\{1,2\}) = \emptyset,\ \varphi(3) = 2.$$

φ-stability of f_I means that

$$H_{\{1,2\}}(f_I \| f_{\{1,2\}}) \le H_{\{1,2\}}(f_I)$$

for all $f_{\{1,2\}} \in S_{\{1,2\}}$,

and

$$H_3(f_I \| (f_2, f_3)) \le H_3(f_I \| f_2)$$

for all $f_2 \in S_2$ and $f_3 \in S_3$.

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Mixed Strategies and Situations

In non-cooperative games probability measures on sets of strategies S_i induce probability measure on the set of situations S_I.

Problem for coalitional games.

Example

$I = \{1, 2, 3\}$, $S_1 = S_2 = S_3 = \{0, 1\}$.
For $\{1, 2\}$ let $P(s_1 = 1 \wedge s_2 = 1) = P(s_1 = 0 \wedge s_2 = 0) = 1/2$,
For $\{1, 3\}$ let $P(s_1 = 1 \wedge s_3 = 1) = P(s_1 = 0 \wedge s_3 = 0) = 1/2$,
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Can't assign any probability to $s_1 = 0 \wedge s_2 = 0 \wedge s_3 = 0$.

Problem of existence of a multidimensional random variable with the given multidimensional projections. Related to "Bell's theorem" in quantum mechanics.

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Regular Simplicial Complexes

Let $\mathcal{K}$ be a (closed) simplicial complex.

Definition

- A maximal face of $\mathcal{K}$ is an extreme face if it has at least one proper vertex (i.e. not belonging to another maximal face).
- Let T be an extreme face of $\mathcal{K}$. The subcomplex obtained from $\mathcal{K}$ by removing all proper vertices of T with their stars is called normal subcomplex of $\mathcal{K}$.
- $\mathcal{K}$ is regular if there is a sequence

$$\mathcal{K} \supset \mathcal{K}_1 \supset \cdots \supset \mathcal{K}_n$$

of subcomplexes such that $\mathcal{K}_i$ is a normal subcomplex of $\mathcal{K}_{i-1}$ and $\mathcal{K}_n = \emptyset$.

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Regular Simplicial Complexes

Let $\mathcal{K}$ be a (closed) simplicial complex.

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- A maximal face of $\mathcal{K}$ is an extreme face if it has at least one proper vertex (i.e. not belonging to another maximal face).
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Extension of Mixed Strategies

Example

- The union of a simplex and all its faces is a regular complex.
- The same minus the simplex itself is a complex which is not regular.

Theorem

The regularity of the complex $\mathcal{K}_{ac}$ is necessary and sufficient for any compatible family of mixed strategies of coalitions in $\mathcal{K}_{ac}$ to have an extension to a probability measure on S_I.

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φ^*-stability

Define the following class of coalitional games.

Let $\mathcal{K}_{ac}$ be an arbitrary regular simplicial complex with the set of vertices I, and

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be an arbitrary normal sequence.

Let T_i be an extreme face of $\mathcal{K}_i$ and $Q_i = T_i \setminus |\mathcal{K}_{i-1}|$ be its set of proper vertices.

For each Q_i consider a partition $Q_i = K_{i1} \cup \cdots \cup K_{ir_i}$.

Take $\{K_{ij}\}_{1 \leq i \leq n,\ 1 \leq j \leq r_i}$ as $\mathcal{K}_{in}$.

Define the map $\varphi^* : \mathcal{K}_{in} \to \mathcal{K}_{ac}$ inductively:

- $\varphi^*(K_{11}) = \emptyset$,
- $\varphi^*(K_{ij}) \subset \bigcup_{k \leq i,\ \ell \leq j} K_{k\ell}$ with $K_{ij} \cup \varphi^*(K_{ij}) \in \mathcal{K}_{ac}$ and $K_{ij} \cap \varphi^*(K_{ij}) = \emptyset$.

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The situation μ (in mixed coalitional strategies) is φ^*-stable if

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for all $K_{ij} \in \mathcal{K}_{in}, \; f_{K_{ij}} \in S_{K_{ij}}, \; f_{\varphi^*(K_{ij})} \in S_{\varphi^*(K_{ij})}$.

Theorem

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