Cutting a Ball in Two

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Decomposition of definable sets into topological cells.

Definable (in o-minimal structure), e.g., semialgebraic or subanalytic.

Topological n-cell $=$ homeomorphic image of a standard n-ball, $\{(x_1, \dots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 < 1\}$ or $(-1, 1)^n$.

An n-cell B^n is *regular* if $(\overline{B^n},\ B^n)$ is homeomorphic (as a pair) to the standard pair $([-1, 1],\ (-1, 1))$.

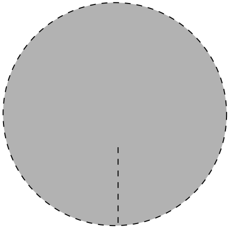

Figure: Example of non-regular cell

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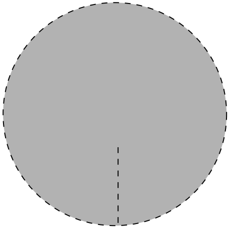

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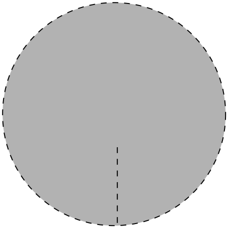

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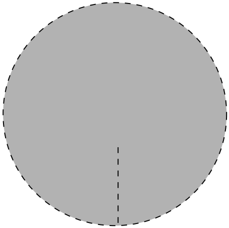

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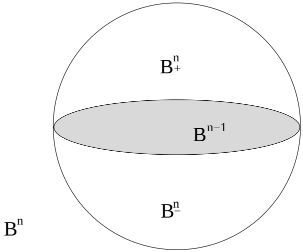

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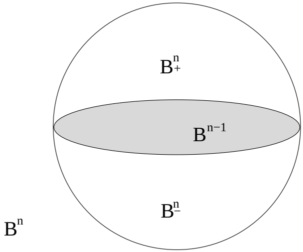

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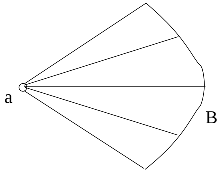
Polyhedra and PL maps.

Definition

Let $a \in \mathbb{R}^n$, $B \subset \mathbb{R}^n$. The subset aB is a *cone* with vertex a and base B if each point in aB is expressed uniquely as $\lambda a + (1 - \lambda)b$ for some $b \in B$ and $0 \le \lambda \le 1$.



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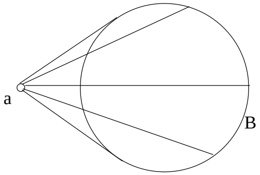

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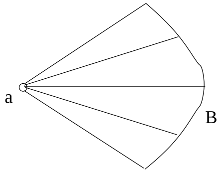

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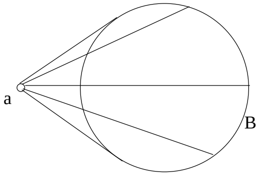

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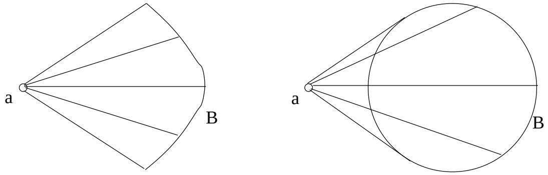

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A subset $P \subset \mathbb{R}^n$ is a *polyhedron* if every point $a \in P$ has a cone neighbourhood (called *star*) aB in P, where B is compact.

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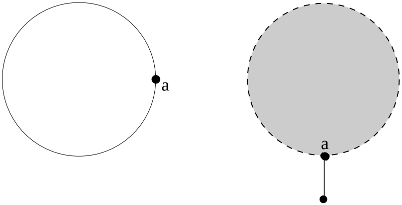
1. Any (geometric realization of) simplicial complex.
2. Any open set in $\mathbb{R}^n$.
3. An intersection of finitely many polyhedra.
4. Non-polyhedra:



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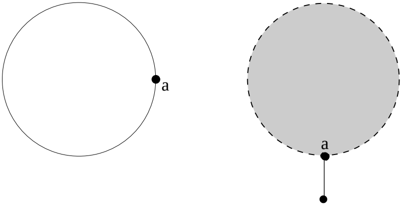

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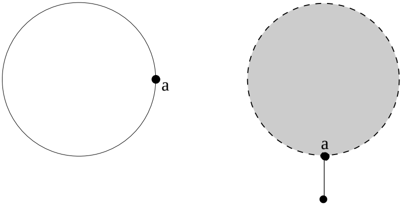

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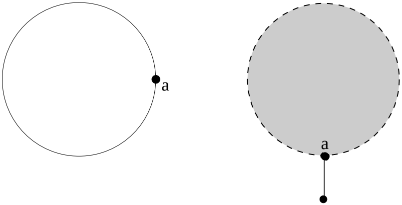

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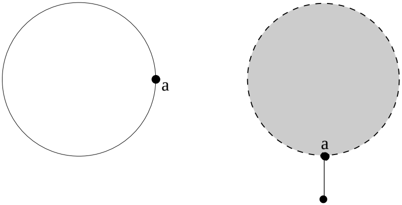

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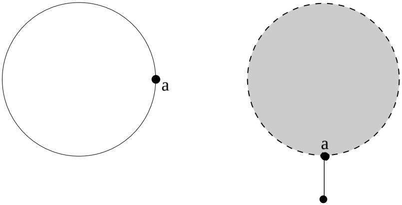

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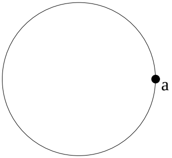

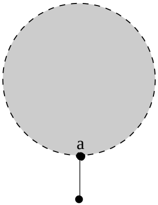

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A map $f: \ P \to Q$ is called *piecewise linear (PL)* if every point $a \in P$ has a star aB such that $f(\lambda a + (1 - \lambda)b) = \lambda f(a) + (1 - \lambda)f(b)$ for all $b \in B$ and $0 \leq \lambda \leq 1$.

Obviously, a linear map is PL.

Exercise

A map $f: \ P \to Q$ is PL iff its graph $\{(x, f(x)) \in \mathbb{R}^{n+m} | \ x \in P\}$ is a polyhedron.

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Every compact definable set is *triangulable*, i.e., is definably homeomorphic to a geometric realization of a finite simplicial complex, i.e., to a polyhedron.
Open definable sets are polyhedra.
Every PL map (or homeomorphism) is a definable map (or homeomorphism).
Hence, certain questions about homeomorphisms of definable sets can be reduced to PL homeomorphisms of corresponding polyhedra.

The converse is also partly true, o-minimal **Hauptvermutung**: *Two definably homeomorphic compact polyhedra are PL homeomorphic.*

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More generally, the Hauptvermutung is false (Milnor, 1960, dim 6 polyhedron).

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$B^n \subset \mathbb{R}^k$ is a closed (resp. open) *PL n-ball* if B^n is PL homeomorphic to $[-1,1]^n$ (resp. $(-1,1)^n$). (An open n-ball is, of course an n-cell.)
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Let B and D be n-balls, and $h : \partial B \to \partial D$ a homeomorphism. Then h extends to a homeomorphism $h' : B \to D$.

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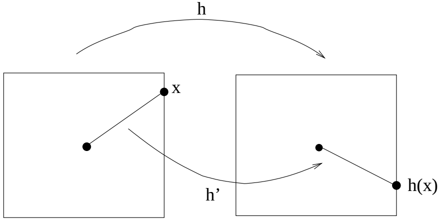

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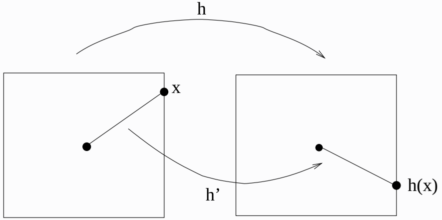

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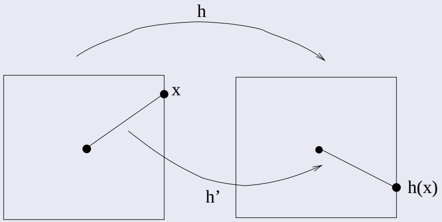

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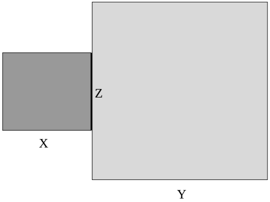

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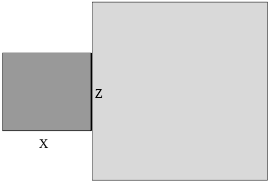

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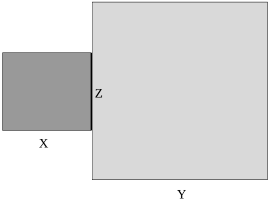

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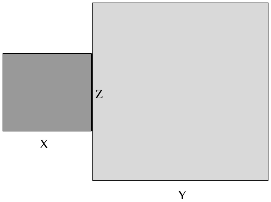

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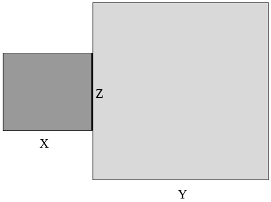

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Converse?

Lemma (Shiota)

Let $X, Y \subset \mathbb{R}^n$ be compact polyhedra such that X and $X \cup Y$ are closed n-balls. Let $X \cap Y$ be a closed $(n-1)$-ball in ∂X, and let $int(X \cap Y) \subset int(X \cup Y)$. Then Y is a closed ball.

We want to get rid of the assumption that X is a ball.

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Jordan theorem: $S^1 \subset \mathbb{R}^2$ "divides" $\mathbb{R}^2$ into two connected components.
Same for $S^1 \subset S^2$.

Schönflies theorem: in addition, the components are homeomorphic to $[-1,1]^2$ and *closire*$(\mathbb{R}^2 \setminus [-1,1]^2)$.
(In case $S^1 \subset S^2$, both are homeomorphic to $[-1,1]^2$.)

Jordan theorem generalizes to all higher dimensions: if $S^{n-1} \to S^n$ is an embedding then S^{n-1} divides S^n into two parts (a very special case of Alexander duality).
But direct generalization of Schönflies is false already for $n = 3$: the Alexander Horned sphere.

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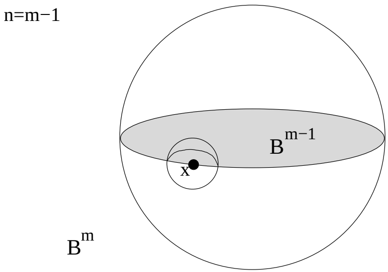
Photograph by Claire Ferguson

PLATE 3. Alexander horned wild sphere, patina bronze, 9" diameter, by Helaman Ferguson

Definition

A pair of (PL) balls (B^m, B^n) is *proper* if $B^n \cap \partial B^m = \partial B^n$. A proper pair is *locally flat* if each point $x \in B^n$ has a neighbourhood in (B^m, B^n) homeomorphic (as a pair) to $(\mathbb{R}^m_+, \mathbb{R}^n_+ \times 0)$ for $x \in \partial B^m$ and to $(\mathbb{R}^m, \mathbb{R}^n \times 0)$ otherwise. Similarly for spheres (S^m, S^n).



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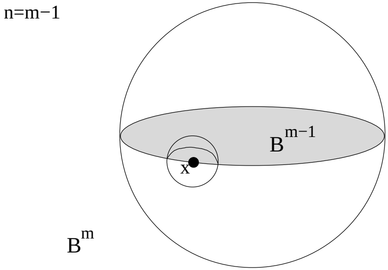

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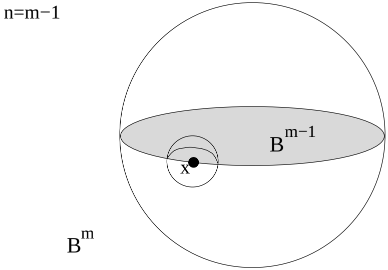

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Theorem (Generalized Schönflies theorem)

For $n \neq 4$ any locally flat pair of PL spheres (S^n, S^{n-1}) is unknotted.

Same in **Diff** category. In **Top** Schönflies is true for every n (Brown, Mazur and Morse, 1960)

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For $n \neq 4, 5$ any locally flat pair of PL balls (B^n, B^{n-1}) is unknotted.

Proof (communicated by N. Mnev).

Notation:

$S^{n-1} := \partial B^n$, $S^{n-2} := \partial B^{n-1}$

B^n_+, B^n_- are two connected components of B^n separated by B^{n-1}

S^{n-1}_+, S^{n-1}_- are corresponding components of S^{n-1} □

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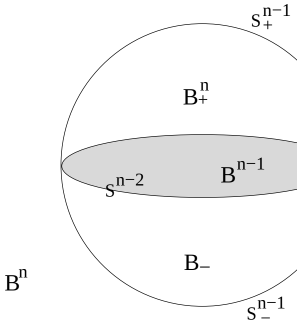

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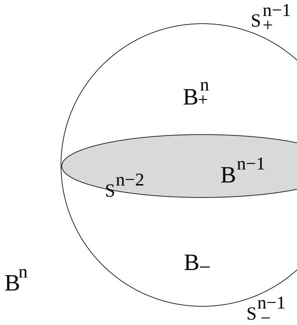

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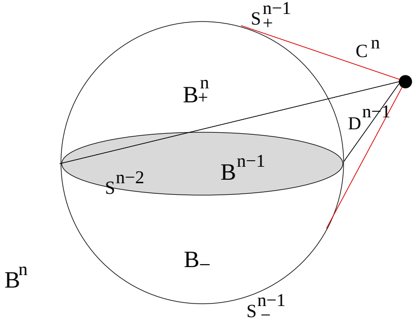

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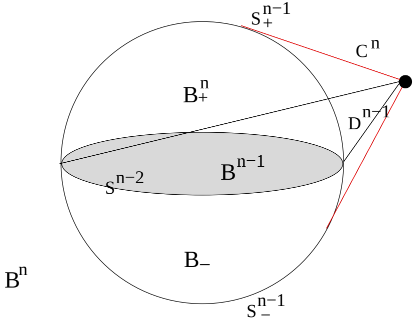

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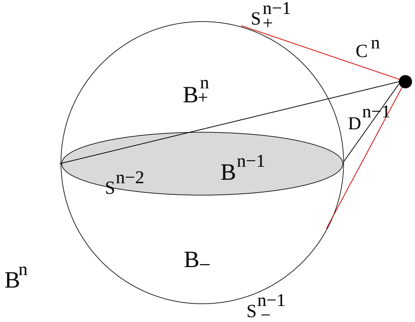

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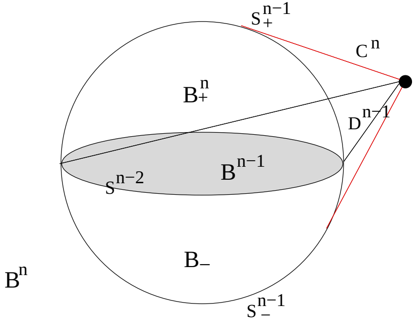

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These two parts are:

$B_+^n \cup E_+^n$ and $B_-^n \cup E_-^n$
where E_+^n and E_-^n are cones with bases S_+^{n-1} and S_-^{n-1} respectively.

S_+^{n-1} and S_-^{n-1} are PL $(n-1)$-balls. Hence E_+^n and E_-^n are PL n-balls.
Shiota's lemma implies that B_+^n and B_-^n are PL n-balls.

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These two parts are:

$B^n_+ \cup E^n_+$ and $B^n_- \cup E^n_-$
where E^n_+ and E^n_- are cones with bases S^{n-1}_+ and S^{n-1}_- respectively.

S^{n-1}_+ and S^{n-1}_- are PL $(n-1)$-balls. Hence E^n_+ and E^n_- are PL n-balls.
Shiota's lemma implies that B^n_+ and B^n_- are PL n-balls.

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Shiota's lemma implies that B^n_+ and B^n_- are PL n-balls.

Photograph by Claire Ferguson

PLATE 1. Tame sphere, Inner Mongolian black granite, 16" diameter, by Helaman Ferguson