Cutting a Ball in Two

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Decomposition of definable sets into topological cells.
Definable (in o-minimal structure), e.g., semialgebraic or subanalytic.
Topological $n$-cell $= \text{homeomorphic image of a standard } n\text{-ball},$
$\{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 < 1\}$ or $(-1, 1)^n$.
An $n$-cell $B^n$ is \textit{regular} if $(B^n, B^n)$ is homeomorphic (as a pair) to the standard pair $([-1, 1], (-1, 1))$.

\textbf{Figure:} Example of non-regular cell
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**Figure:** Example of non-regular cell
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**Question:** Under which conditions parts $B_+^n$ and $B_-^n$ are regular cells?
Definition

Let $a \in \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$. The subset $aB$ is a cone with vertex $a$ and base $B$ if each point in $aB$ is expressed uniquely as $\lambda a + (1 - \lambda)b$ for some $b \in B$ and $0 \leq \lambda \leq 1$. 

![Diagram](image-url)

- **cone**
- **not cone**
 PIECEWISE-LINEAR (PL) TOPOLOGY  

Polyhedra and PL maps.

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**Diagram:**

- **Cone:** $aB$ where each point in $aB$ can be uniquely expressed as $\lambda a + (1 - \lambda)b$ for some $b \in B$ and $0 \leq \lambda \leq 1$.
- **Not Cone:** $aB$ where the condition for a cone is not satisfied.
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Definition

A subset $P \subset \mathbb{R}^n$ is a *polyhedron* if every point $a \in P$ has a cone neighbourhood (called *star*) $aB$ in $P$, where $B$ is compact.

Example

1. Any (geometric realization of) simplicial complex.
2. Any open set in $\mathbb{R}^n$.
3. An intersection of finitely many polyhedra.
4. Non-polyhedra:
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**Definition**

A map \( f : P \to Q \) is called *piecewise linear (PL)* if every point \( a \in P \) has a star \( aB \) such that

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f(\lambda a + (1 - \lambda)b) = \lambda f(a) + (1 - \lambda)f(b)
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for all \( b \in B \) and \( 0 \leq \lambda \leq 1 \).

Obviously, a linear map is PL.

**Exercise**

A map \( f : P \to Q \) is PL iff its graph \( \{(x, f(x)) \in \mathbb{R}^{n+m} | x \in P\} \) is a polyhedron.

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What is the relation to definable sets?

Every compact definable set is *triangulable*, i.e., is definably homeomorphic to a geometric realization of a finite simplicial complex, i.e., to a polyhedron.

Open definable sets are polyhedra.

Every PL map (or homeomorphism) is a definable map (or homeomorphism).

Hence, certain questions about homeomorphisms of definable sets can be reduced to PL homeomorphisms of corresponding polyhedra.

The converse is also partly true, o-minimal Hauptvermutung: *Two definably homeomorphic compact polyhedra are PL homeomorphic.*

M. Shiota, Geometry of Subanalytic and Semialgebraic Sets, Birkhäuser, 1997

More generally, the Hauptvermutung is false (Milnor, 1960, dim 6 polyhedron).
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$M$ is an $n$-manifold with boundary if every point has a neighbourhood homeomorphic to an open subset of either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. The boundary $\partial M$ of $M$ is an (unbounded) $(n-1)$-manifold consisting of points corresponding to $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n_+$. 

$B^n \subset \mathbb{R}^k$ is a closed (resp. open) PL $n$-ball if $B^n$ is PL homeomorphic to $[-1, 1]^n$ (resp. $(-1, 1)^n$). (An open $n$-ball is, of course an $n$-cell.) $S^n \subset \mathbb{R}^k$ is a PL $n$-sphere if $S^n$ is PL homeomorphic to $\partial[-1, 1]^{n+1}$. 

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The interior $B^n \setminus \partial B^n$ of any closed ball $B^n$ is a regular cell:

**Lemma**

Let $B$ and $D$ be $n$-balls, and $h : \partial B \to \partial D$ a homeomorphism. Then $h$ extends to a homeomorphism $h' : B \to D$.

**Proof.**
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**Proof.**

[Diagram showing the extension of $h$ to $h'$]
Definition

Let $Z$ be a closed (open) $(n - 1)$-ball, $X$, $Y$ be closed (resp., open) $n$-balls, and $Z = X \cap Y = \partial X \cap \partial Y$. We say that $X \cup Y \cup Z$ is obtained by *gluing* $X$ and $Y$ along $Z$.

Theorem

If $X$, $Y$, $Z$ are closed balls, then $X \cup Y$ is a closed ball.
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Converse?

**Lemma (Shiota)**

Let $X, Y \subset \mathbb{R}^n$ be compact polyhedra such that $X$ and $X \cup Y$ are closed $n$-balls. Let $X \cap Y$ be a closed $(n - 1)$-ball in $\partial X$, and let $\text{int}(X \cap Y) \subset \text{int}(X \cup Y)$. Then $Y$ is a closed ball.

We want to get rid of the assumption that $X$ is a ball.

Start with spheres
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Let $X, Y \subset \mathbb{R}^n$ be compact polyhedra such that $X$ and $X \cup Y$ are closed $n$-balls. Let $X \cap Y$ be a closed $(n-1)$-ball in $\partial X$, and let $\text{int}(X \cap Y) \subset \text{int}(X \cup Y)$. Then $Y$ is a closed ball.

We want to get rid of the assumption that $X$ is a ball.

Start with spheres
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SCHÖNFLIES THEOREM

Jordan theorem: $S^1 \subset \mathbb{R}^2$ “divides” $\mathbb{R}^2$ into two connected components.
Same for $S^1 \subset S^2$.

Schönflies theorem: in addition, the components are homeomorphic to $[-1, 1]^2$ and $\text{closure}(\mathbb{R}^2 \setminus [-1, 1]^2)$.
(In case $S^1 \subset S^2$, both are homeomorphic to $[-1, 1]^2$.)

Jordan theorem generalizes to all higher dimensions: if $S^{n-1} \to S^n$ is an embedding then $S^{n-1}$ divides $S^n$ into two parts (a very special case of Alexander duality).
But direct generalization of Schönflies is false already for $n = 3$: the Alexander Horned sphere.

Nicolai Vorobjov
Cutting a Ball in Two
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PLATE 3. Alexander horned wild sphere, patina bronze, 9" diameter, by Helaman Ferguson
Definition

A pair of (PL) balls \((B^m, B^n)\) is proper if \(B^n \cap \partial B^m = \partial B^n\). A proper pair is locally flat if each point \(x \in B^n\) has a neighbourhood in \((B^m, B^n)\) homeomorphic (as a pair) to \((\mathbb{R}_+^m, \mathbb{R}_+^n \times 0)\) for \(x \in \partial B^m\) and to \((\mathbb{R}^m, \mathbb{R}_+^n \times 0)\) otherwise. Similarly for spheres \((S^m, S^n)\).
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\((\mathbb{R}^m_+, \mathbb{R}^n_+ \times 0)\) for \(x \in \partial B^m\) and to 
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Definition

A ball pair \((B^m, B^n)\) is *unknotted* if it is homeomorphic to \((-1, 1]^m \times (-1, 1]^n \times 0\).

A sphere pair \((S^m, S^n)\) is *unknotted* if it is homeomorphic to \((\partial\,[-1, 1]^m, \partial\,[-1, 1]^n \times 0)\).

Theorem (Generalized Schönflies theorem)

For \(n \neq 4\) any locally flat pair of PL spheres \((S^n, S^{n-1})\) is unknotted.

Same in **Diff** category. In **Top** Schönflies is true for every \(n\) (Brown, Mazur and Morse, 1960)
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Theorem

For $n \neq 4, 5$ any locally flat pair of PL balls $(B^n, B^{n-1})$ is unknotted.

Proof (communicated by N. Mnev).

Notation:
$S^{n-1} := \partial B^n, S^{n-2} := \partial B^{n-1}$

$B^n_+, B^n_- \text{ are two connected components of } B^n \text{ separated by } B^{n-1}$

$S^{n-1}_+, S^{n-1}_- \text{ are corresponding components of } S^{n-1}$
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(\(C^n, D^{n-1}\)) cone pair with the base (\(S^{n-1}, S^{n-2}\)). Then \(W^n := B^n \cup C^n\) is an \(n\)-sphere, \(V^{n-1} := B^{n-1} \cup D^{n-1}\) is an \((n - 1)\)-sphere, and the pair \((W^n, V^{n-1})\) is locally flat. By Schönflies theorem, for \(n \neq 4\), \((W^n, V^{n-1})\) is unknotted, and the two parts of \(W^n\) separated by \(V^{n-1}\) are PL balls.
(C^n, D^{n-1}) cone pair with the base (S^{n-1}, S^{n-2}). Then W^n := B^n \cup C^n is an n-sphere, V^{n-1} := B^{n-1} \cup D^{n-1} is an (n − 1)-sphere, and the pair (W^n, V^{n-1}) is locally flat. By Schönflies theorem, for n \neq 4, (W^n, V^{n-1}) is unknotted, and the two parts of W^n separated by V^{n-1} are PL balls.
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These two parts are:

\[ B^n_+ \cup E^n_+ \text{ and } B^n_- \cup E^n_- \]

where \( E^n_+ \) and \( E^n_- \) are cones with bases \( S^{n-1}_+ \) and \( S^{n-1}_- \) respectively.

\( S^{n-1}_+ \) and \( S^{n-1}_- \) are PL \((n-1)\)-balls. Hence \( E^n_+ \) and \( E^n_- \) are PL \( n \)-balls.

Shiota’s lemma implies that \( B^n_+ \) and \( B^n_- \) are PL \( n \)-balls.
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\[ B^n_+ \cup E^n_+ \text{ and } B^n_- \cup E^n_- \]

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Shiota’s lemma implies that \( B^n_+ \) and \( B^n_- \) are PL \(n\)-balls.
These two parts are:

\[ B_+^n \cup E_+^n \text{ and } B_-^n \cup E_-^n \]

where \( E_+^n \) and \( E_-^n \) are cones with bases \( S_+^{n-1} \) and \( S_-^{n-1} \) respectively.

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Shiota’s lemma implies that \( B^+_n \) and \( B^-_n \) are PL \(n\)-balls.
Plate 1. Tame sphere, Inner Mongolian black granite, 16" diameter, by Helaman Ferguson

Photograph by Claire Ferguson

Nicolai Vorobjov

Cutting a Ball in Two