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Approximation of definable sets by compact families, and upper bounds on homotopy and homology

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Abstract

We prove new upper bounds on homotopy and homology groups of o-minimal sets in terms of their approximations by compact o-minimal sets. In particular, we improve the known upper bounds on Betti numbers of semialgebraic sets defined by quantifier-free formulae and obtain, for the first time, a singly exponential bound on Betti numbers of sub-Pfaffian sets.

Introduction

We study upper bounds on the topological complexity of sets definable in o-minimal structures over the reals. The fundamental case of algebraic sets in $\mathbb{R}^n$ was first considered in around 1950 by Petrovskii and Oleinik [**13**, **14**], and then in the 1960s by Milnor [**12**] and Thom [**17**]. They gave explicit upper bounds on total Betti numbers in terms of degrees and numbers of variables of the defining polynomials.

There are two natural approaches to generalizing and expanding these results. First, noticing that not much of algebraic geometry is used in the proofs, one can obtain similar upper bounds for polynomials with the 'description complexity' measure when that is different from the degree, and for non-algebraic functions, such as Khovanskii's fewnomials and Pfaffian functions [**11**]. A bound for algebraic sets defined by quadratic polynomials was proved in [**1**].

Second, the bounds can be expanded to semialgebraic and semi-Pfaffian sets defined by formulae more general than just conjunctions of equations. Basu [**2**] proved an asymptotically tight upper bound on Betti numbers in the case of semialgebraic sets defined by conjunctions and disjunctions of non-strict inequalities. The proof can easily be extended to special classes of non-algebraic functions. For fewnomials and Pfaffian functions, this was done by Zell [**19**]. For quadratic polynomials an upper bound was proved in [**3**]. The principal difficulty arises when neither the set itself nor its complement is locally closed.

Until recently, the best available upper bound for the Betti numbers of a semialgebraic set defined by an arbitrary Boolean combination of equations and inequalities remained doubly exponential in the number of variables. The first singly exponential upper bound was obtained by the authors in [**9**] based on a construction that replaces a given semialgebraic set by a homotopy equivalent compact semialgebraic set. This construction extends to semi-Pfaffian sets and, more generally, to the sets defined by Boolean combinations of equations and inequalities between continuous functions definable in an o-minimal structure over $\mathbb{R}$. It cannot be applied to sets defined by formulae with quantifiers, such as sub-Pfaffian sets, but can be used in conjunction with effective quantifier elimination in the semialgebraic situation.

In [**10**] we obtained a spectral sequence converging to the homology of the projection of an o-minimal set under the closed continuous surjective definable map. It gives an upper bound on the Betti number of the projection that, in the semialgebraic case, is better than the one

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based on quantifier elimination. The requirement for the map to be closed can be relaxed but not completely removed, which left the upper bound problem unresolved in the general Pfaffian case, where quantifier elimination is not applicable.

In this paper we suggest a new construction for approximating a large class of definable sets, including the sets defined by arbitrary Boolean combinations of equations and inequalities, by compact sets. The construction is applicable to images of such sets under a large class of definable maps, for example, projections. Based on this construction, we refine the results from [**9**, **10**] and prove similar upper bounds, individually for different Betti numbers, for images under arbitrary continuous definable maps.

In the semialgebraic case the bound from [**9**] is the square of the number of different polynomials occurring in the formula, while the bounds proved in this paper multiply the number of polynomials by a typically smaller coefficient that does not exceed the dimension. This is especially relevant for applications to problems of subspace arrangements, robotics, and visualization, where the dimension and degrees usually remain small, while the number of polynomials is very large. Applied to projections, the bounds are stronger than the ones obtained by the effective quantifier elimination.

In the non-algebraic case, for the first time, the bounds, singly exponential in the number of variables, are obtained for projections of semi-Pfaffian sets, as well as projections of sets defined by Boolean formulae with polynomials from special classes.

Notation. In this paper we use the following (standard) notation. For a topological space X, we denote by $H_i(X)$ its singular homology group with coefficients in some fixed Abelian group, by $\pi_i(X)$ we denote the homotopy group (provided that X is connected), the symbol $\simeq$ denotes the homotopy equivalence, and the symbol $\cong$ stands for the group isomorphism. If $Y \subset X$, then $\overline{Y}$ denotes its closure in X.

1. *Main result*

In what follows we fix an o-minimal structure over $\mathbb{R}$ and consider sets, families of sets, maps, etc. that are *definable* in this structure.

Definition 1.1. Let G be a definable compact set. Consider a definable family $\{S_\delta\}_{\delta>0}$ of compact subsets of G such that, for all $\delta', \delta \in (0,1)$, if $\delta' > \delta$, then $S_{\delta'} \subset S_\delta$. We write $S := \bigcup_{\delta>0} S_\delta$.

For each $\delta > 0$, let $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ be a definable family of compact subsets of G such that the following hold:

(i) for all $\varepsilon, \varepsilon' \in (0,1)$, if $\varepsilon' > \varepsilon$, then $S_{\delta,\varepsilon} \subset S_{\delta,\varepsilon'}$;
(ii) $S_\delta = \bigcap_{\varepsilon>0} S_{\delta,\varepsilon}$;
(iii) for all $\delta' > 0$ sufficiently smaller than δ, and for all $\varepsilon' > 0$, there exists an open in G set $U \subset G$ such that $S_\delta \subset U \subset S_{\delta',\varepsilon'}$.

We say that S is *represented* by the families $\{S_\delta\}_{\delta>0}$ and $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G.

Let S' be represented by $\{S'_\delta\}_{\delta>0}$ and $\{S'_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G and let S'' be represented by $\{S''_\delta\}_{\delta>0}$ and $\{S''_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G.

Lemma 1.2. *The set $S' \cap S''$ is represented by the families $\{S'_\delta \cap S''_\delta\}_{\delta>0}$ and $\{S'_{\delta,\varepsilon} \cap S''_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G, while $S' \cup S''$ is represented by $\{S'_\delta \cup S''_\delta\}_{\delta>0}$ and $\{S'_{\delta,\varepsilon} \cup S''_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G.*

The proof of Lemma 1.2 follows from a straightforward checking of Definition 1.1.

Let S be represented by $\{S_\delta\}_{\delta>0}$ and $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G and let $F : D \to H$ be a continuous definable map, where D and H are definable, $S \subset D \subset G$, and H is compact.

Lemma 1.3. *Let D be open in G, and F be an open map. Then $F(S)$ is represented by the families $\{F(S_\delta)\}_{\delta>0}$ and $\{F(S_{\delta,\varepsilon})\}_{\delta,\varepsilon>0}$ in H.*

The proof follows from a straightforward checking of Definition 1.1 (openness is required for (iii) to hold).

Consider the projections $\rho_1 : G \times H \to G$ and $\rho_2 : G \times H \to H$. Let $\Gamma \subset G \times H$ be the graph of F. Suppose that Γ is represented by the families $\{\Gamma_\delta\}_{\delta>0}$ and $\{\Gamma_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in $G \times H$.

Lemma 1.4. *The set $F(S)$ is represented by the families*

$$\{\rho_2(\rho_1^{-1}(S_\delta) \cap \Gamma_\delta)\}_{\delta>0} \quad \textit{and} \quad \{\rho_2(\rho_1^{-1}(S_{\delta,\varepsilon}) \cap \Gamma_{\delta,\varepsilon})\}_{\delta,\varepsilon>0}$$

in H.

Proof. The set $\rho_1^{-1}(S)$ is represented by the families

$$\{\rho_1^{-1}(S_\delta) \cap \Gamma_\delta\}_{\delta>0} \quad \text{and} \quad \{\rho_1^{-1}(S_{\delta,\varepsilon}) \cap \Gamma_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$$

in $G \times H$, and the projection ρ_2 satisfies Lemma 1.3. $\square$

Along with this general case, we will be considering the following important particular cases.

Let $S = \{\mathbf{x} \mid \mathcal{F}(\mathbf{x})\} \subset \mathbb{R}^n$ be a bounded definable set of points satisfying a Boolean combination $\mathcal{F}$ of equations of the kind $h(\mathbf{x}) = 0$ and inequalities of the kind $h(\mathbf{x}) > 0$, where $h : \mathbb{R}^n \to \mathbb{R}$ are continuous definable functions (for example, polynomials). Here we let G be a closed ball of a sufficiently large radius centred at 0. We now define the representing families $\{S_\delta\}$ and $\{S_{\delta,\varepsilon}\}$.

Definition 1.5. For a given finite set $\{h_1, \dots, h_k\}$ of functions $h_i : \mathbb{R}^n \to \mathbb{R}$, define its *sign set* as a non-empty subset in $\mathbb{R}^n$ of the kind

$$h_{i_1} = \ldots = h_{i_{k_1}} = 0, \quad h_{i_{k_1+1}} > 0, \dots, h_{i_{k_2}} > 0, \quad h_{i_{k_2+1}} < 0, \dots, h_{i_k} < 0,$$

where $i_1, \dots, i_{k_1}, \dots, i_{k_2}, \dots, i_k$ is a permutation of $1, \dots, k$.

Let $\{h_1, \dots, h_k\}$ be the set of all functions in the Boolean formula defining S. Then S is a disjoint union of some sign sets of $\{h_1, \dots, h_k\}$. The set S_δ is the result of the replacement, independently in each sign set in this union, of all inequalities $h > 0$ and $h < 0$ by $h \geqslant \delta$ and $h \leqslant -\delta$, respectively. The set $S_{\delta,\varepsilon}$ is obtained by replacing, independently in each sign set, all expressions $h > 0$, $h < 0$, and $h = 0$ by $h \geqslant \delta$, $h \leqslant -\delta$, and $-\varepsilon \leqslant h \leqslant \varepsilon$, respectively. According to Lemma 1.2, the set S, being the union of sign sets, is represented by the families $\{S_\delta\}$ and $\{S_{\delta,\varepsilon}\}$ in G.

Example 1.6. Let the closed quadrant S be defined as the union of sign sets $\{x > 0, y > 0\} \cup \{x > 0,\ y = 0\} \cup \{x = 0,\ y > 0\} \cup \{x = y = 0\}$. Figure 1 shows the corresponding set $S_{\delta,\varepsilon}$ for $\varepsilon < \delta$.

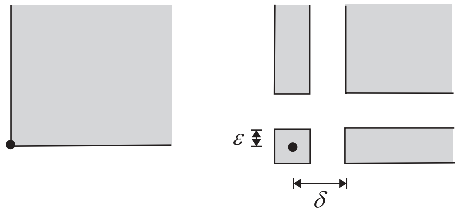

FIGURE 1. *The set* $S_{\delta,\varepsilon}$ (*right*) *for the closed quadrant* S (*left*).

Now suppose that the set $S \subset \mathbb{R}^n$, defined as above by a Boolean formula $\mathcal{F}$, is not necessarily bounded. In this case we let G be the definable one-point (Alexandrov) compactification of $\mathbb{R}^n$. Note that each function h is continuous in $G \setminus \{\infty\}$. Define sets S_δ and $S_{\delta,\varepsilon}$ as in the bounded case, replacing equations and inequalities, independently in each sign set of $\{h_1, \dots, h_k\}$, and then taking the conjunction of the resulting formula with $|\mathbf{x}|^2 \leqslant 1/\delta$. Again, S is represented by $\{S_\delta\}$ and $\{S_{\delta,\varepsilon}\}$ in G, and in what follows we will refer to this instance as the *constructible case*.

DEFINITION 1.7. Let $\mathcal{P} := \mathcal{P}(\varepsilon_0, \dots, \varepsilon_\ell)$ be a predicate (property) over $(0,1)^{\ell+1}$. We say that the property $\mathcal{P}$ holds for

$$0 < \varepsilon_0 \ll \varepsilon_1 \ll \dots \ll \varepsilon_\ell \ll 1$$

if there exist definable functions $f_k : (0,1)^{\ell-k} \to (0,1)$, where $k = 0, \dots, \ell$ (with f_ℓ being a positive constant), such that $\mathcal{P}$ holds for any sequence $\varepsilon_0, \dots, \varepsilon_\ell$ satisfying

$$0 < \varepsilon_k < f_k(\varepsilon_{k+1}, \dots, \varepsilon_\ell) \quad \text{for } k = 0, \dots, \ell.$$

Now we return to the general case in Definition 1.1, which we will refer to, in what follows, as the *definable case*.

DEFINITION 1.8. For a sequence $\varepsilon_0, \delta_0, \varepsilon_1, \delta_1, \dots, \varepsilon_m, \delta_m$, where $m \geqslant 0$, introduce the compact set

$$T(S) := S_{\delta_0,\varepsilon_0} \cup S_{\delta_1,\varepsilon_1} \cup \dots \cup S_{\delta_m,\varepsilon_m}.$$

From Definition 1.1 it is easy to see that, for any $m \geqslant 0$ and for

$$0 < \varepsilon_0 \ll \delta_0 \ll \varepsilon_1 \ll \delta_1 \ll \dots \ll \varepsilon_m \ll \delta_m \ll 1, \tag{1.1}$$

there is a surjective map $C : \mathbf{T} \to \mathbf{S}$ from the finite set $\mathbf{T}$ of all connected components of $T(S)$ onto the set $\mathbf{S}$ of all connected components of S such that, for any $S' \in \mathbf{S}$, we have

$$\bigcup_{T' \in C^{-1}(S')} T' = T(S').$$

LEMMA 1.9. *If* $m > 0$ *then* C *is bijective.*

Proof. Let S be connected and $m > 0$. We prove that $T(S)$ is connected. Let $\mathbf{x}, \mathbf{y} \in S_{\delta_i,\varepsilon_i} \subset T(S)$. Let $\mathbf{x}_\varepsilon$ and $\mathbf{y}_\varepsilon$ be the definable connected curves such that $\mathbf{x}_{\varepsilon_i} = \mathbf{x}$, $\mathbf{x}_0 := \lim_{\varepsilon \searrow 0} \mathbf{x}_\varepsilon \in S_{\delta_i}$, $\mathbf{y}_{\varepsilon_i} = \mathbf{y}$, and $\mathbf{y}_0 := \lim_{\varepsilon \searrow 0} \mathbf{y}_\varepsilon \in S_{\delta_i}$. Let $\Gamma \subset S$ be a connected compact definable curve containing $\mathbf{x}_0$ and $\mathbf{y}_0$. If Γ is represented by the families $\{S_\delta \cap \Gamma\}$ and $\{S_{\delta,\varepsilon} \cap \Gamma\}$ then

$T(\Gamma) \subset T(S)$. It is easy to see that, under the condition $m > 0$, the one-dimensional set $T(\Gamma)$ is connected. It follows that $\mathbf{x}$ and $\mathbf{y}$ belong to a connected definable curve in $T(S)$. □

In what follows we denote $T := T(S)$ and let $m > 0$. We assume that S is connected in order to make the homotopy groups $\pi_k(S)$ and $\pi_k(T)$ independent of a base point.

THEOREM 1.10. (i) *For* (1.1) *and every* $1 \leqslant k \leqslant m$, *there are epimorphisms*

$$\psi_k : \pi_k(T) \longrightarrow \pi_k(S),$$
$$\varphi_k : H_k(T) \longrightarrow H_k(S),$$

and, in particular, $\operatorname{rank} H_k(S) \leqslant \operatorname{rank} H_k(T)$.

(ii) *In the constructible case, for* (1.1) *and every* $1 \leqslant k \leqslant m-1$, *there are isomorphisms* ψ_k *and* φ_k, *and, in particular,* $\operatorname{rank} H_k(S) = \operatorname{rank} H_k(T)$. *Moreover, if* $m \geqslant \dim(S)$, *then* $T \simeq S$.

The plan of the proof of Theorem 1.10 is as follows. We consider a simplicial complex R in $\mathbb{R}^n$ such that it is a triangulation of G, and S is a union of some open simplices of R. For any sequence $\varepsilon_0, \delta_0, \varepsilon_1, \delta_1, \ldots, \varepsilon_m, \delta_m$, we construct a subset V of the complex R, which is a combinatorial analogy of T, and prove that there are isomorphisms of k-homotopy groups of V and S for $k \leqslant m-1$ and an epimorphism for $k = m$. We prove the same for homology groups. We then show that, for (1.1), there are epimorphisms $\psi_k : \pi_k(T) \to \pi_k(V)$ and $\varphi_k : H_k(T) \to H_k(V)$ for every $k \leqslant m$. We prove that, if the pair $(R, \{S_\delta\}_{\delta>0})$ satisfies a certain 'separability' property (Definition 5.7), then ψ_k and φ_k are isomorphisms for every $k < m$. In particular, in the constructible case, $(R, \{S_\delta\}_{\delta>0})$ is always separable. This completes the proof.

REMARK 1.11. We conjecture that, in the *definable* case, the statement (ii) of Theorem 1.10 is also true, that is, for (1.1) and every $1 \leqslant k \leqslant m-1$, the homomorphisms ψ_k and φ_k are isomorphisms, and $T \simeq S$ when $m \geqslant \dim(S)$.

2. *Topological background*

In this section we formulate some topological definitions and statements that we will use in further proofs.

Recall that a continuous map between topological spaces $f : X \to Y$ is called a *weak homotopy equivalence* if, for every $j > 0$, the induced homomorphism of homotopy groups $f_{\#j} : \pi_j(X) \to \pi_j(Y)$ is an isomorphism.

THEOREM 2.1 (Whitehead theorem on weak homotopy equivalence [**15**, 7.6.24]). *A map between connected CW-complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Let $f : X \to Y$ be a continuous map between path-connected topological spaces.

THEOREM 2.2 (Whitehead theorem on homotopy and homology [**15**, 7.5.9]). *If* $k > 0$ *is such that the induced homomorphism of homotopy groups* $f_{\#j} : \pi_j(X) \to \pi_j(Y)$ *is an isomorphism for* $j < k$ *and an epimorphism for* $j = k$, *then the induced homomorphism of homology groups* $f_{*j} : H_j(X) \to H_j(Y)$ *is an isomorphism for* $j < k$ *and an epimorphism for* $j = k$.

DEFINITION 2.3 [5]. A map $f : P \to Q$, where P and Q are posets with order relations $\preceq_P$ and $\preceq_Q$, respectively, is called a *poset map* if, for $\mathbf{x}, \mathbf{y} \in P$, we have that $\mathbf{x} \preceq_P \mathbf{y}$ implies that $f(\mathbf{x}) \preceq_Q f(\mathbf{y})$. With a poset P is associated the simplicial complex $\Delta(P)$, called the *order complex*, whose simplices are *chains* (totally ordered subsets) of P. Each poset map f induces the simplicial map $f : \Delta(P) \to \Delta(Q)$.

THEOREM 2.4 [5, Theorem 2]. *Let P and Q be connected posets and $f : P \to Q$ be a poset map. Suppose that the fibre $f^{-1}(\Delta(Q_{\preceq q}))$ is k-connected for all $q \in Q$. Then the induced homomorphism $f_{\#j} : \pi_j(\Delta(P)) \to \pi_j(\Delta(Q))$ is an isomorphism for all $j \leqslant k$ and an epimorphism for $j = k + 1$.*

REMARK 2.5. In the formulation and proof of this theorem in [5] the statement that $f_{\#k+1}$ is an epimorphism is missing. Here is how it follows from the proof of Theorem 2 in [5]. In the proof, a map $g : \Delta^{(k+1)}(Q) \to \Delta(P)$ is defined, where $\Delta^{(k+1)}(Q)$ is the $(k + 1)$-dimensional skeleton of $\Delta(Q)$, such that $f \circ g : \Delta^{(k+1)}(Q) \to \Delta(Q)$ is homotopic to the identity map id. Then the induced homomorphism

$$f_{\#k+1} \circ g_{\#k+1} = (f \circ g)_{\#k+1} = \mathrm{id}_{\#k+1} : \pi_{k+1}(\Delta^{(k+1)}(Q)) \longrightarrow \pi_{k+1}(\Delta(Q))$$

is an epimorphism, since any map of a j-dimensional sphere to $\Delta(Q)$ is homotopic to a map of the sphere to $\Delta^{(j)}(Q)$. It follows that $f_{\#k+1}$ is also an epimorphism.

COROLLARY 2.6 (Vietoris–Begle theorem). *Let X and Y be connected simplicial complexes and $f : X \to Y$ be a simplicial map. Then the following hold.*

(i) *If the fibre $f^{-1}(B)$ is k-connected for every closed simplex B in Y, then the induced homomorphism $f_{\#j} : \pi_j(X) \to \pi_j(Y)$ is an isomorphism for all $j \leqslant k$ and an epimorphism for $j = k + 1$.*

(ii) *If the fibre $f^{-1}(B)$ is contractible, then $X \simeq Y$.*

Proof. (i) Consider the barycentric subdivisions $\widehat{X}$ and $\widehat{Y}$ of complexes X and Y, respectively. Note that $\widehat{X} = \Delta(P)$ and $\widehat{Y} = \Delta(Q)$, where P and Q are *simplex posets* of X and Y, respectively (that is, closed simplices ordered by containment). For a closed simplex $B \in Q$ the subcomplex $\Delta(Q_{\preceq B})$ of $\widehat{Y}$ is the union of all simplices of the barycentric subdivision of B. Now (i) follows from Theorem 2.4.

(ii) Since the fibre $f^{-1}(B)$ is contractible, according to (i), the induced homomorphisms $f_{\#j}$ are isomorphisms for all $j > 0$, and hence, by the Whitehead theorem on weak homotopy equivalence (Theorem 2.1), f induces the homotopy equivalence $X \simeq Y$. □

DEFINITION 2.7. Let Δ be a simplicial complex and X be a topological space. A map C taking simplices B to subspaces $C(B)$ of X is called *carrier* if $C(B) \subset C(K)$ for all simplices B and K in Δ such that B is a subsimplex of K, and a continuous map $f : \Delta \to X$ is *carried* by C if $f(B) \subset C(B)$ for all simplices B in Δ.

THEOREM 2.8 (Carrier lemma [5, Lemma 1]). *Fix $k \geqslant 0$, and let $\Delta^{(k)}$ be the k-skeleton of Δ. Then the following hold.*

(i) *If $C(B)$ is $\dim(B)$-connected for all simplices B in $\Delta^{(k)}$, then every two maps $f, g : |\Delta^{(k)}| \to X$ that are both carried by C are homotopic, $f \sim g$.*

(ii) *If $C(B)$ is $(\dim(B) - 1)$-connected for all simplices B in $\Delta^{(k)}$, then there exists a map $|\Delta^{(k)}| \to X$ carried by C.*

Definition 2.9. The *nerve* of a family $\{X_i\}_{i\in I}$ of sets is the (abstract) simplicial complex $\mathcal{N}$ defined on the vertex set I so that a simplex $\sigma \subset I$ is in $\mathcal{N}$ if and only if $\bigcap_{i\in\sigma} X_i \neq \emptyset$.

Let X be a connected regular CW-complex and $\{X_i\}_{i\in I}$ be a family of its subcomplexes such that $X = \bigcup_{i\in I} X_i$. Let $|\mathcal{N}|$ denote the geometric realization of the nerve $\mathcal{N}$ of $\{X_i\}_{i\in I}$.

Theorem 2.10 (Nerve theorem, [5, Theorem 6]). *The following statements hold.*
(i) *If every non-empty finite intersection $X_{i_1} \cap \ldots \cap X_{i_t}$, where $t \geqslant 1$, is $(k-t+1)$-connected, then there is a map $f : X \to |\mathcal{N}|$ such that the induced homomorphism $f_{\#j} : \pi_j(X) \to \pi_j(|\mathcal{N}|)$ is an isomorphism for all $j \leqslant k$ and an epimorphism for $j = k+1$.*
(ii) *If every non-empty finite intersection $X_{i_1} \cap \ldots \cap X_{i_t}$, where $t \geqslant 1$, is contractible, then $X \simeq |\mathcal{N}|$.*

Remark 2.11. As with Theorem 2.4, in the formulation and proof of this theorem in [5] the statement that $f_{\#k+1}$ is an epimorphism is missing. This statement follows from the proof of Theorem 6 in [5] by the same argument as described in Remark 2.5.

Remark 2.12. Let X be a connected triangulated set, $\{X_i\}_{i\in I}$ be a family of all of its (open) simplices, and the nerve $\mathcal{N}_X$ is defined on the index set I so that a simplex $\sigma \subset I$ is in $\mathcal{N}_X$ if and only if the family $\{X_i\}_{i\in\sigma}$, after a suitable ordering, forms a $|\sigma|$-flag (see Definition 3.1). For this version of the nerve, Theorem 2.10 also holds true. Indeed, it is applicable to the union $X' := \bigcup_i X_i$ of simplices X_i in X that are contained in X with their closures (hence X' may be a proper subset of X). Since $X' \simeq X$ and $|\mathcal{N}_{X'}| \simeq |\mathcal{N}_X|$, Theorem 2.10 is also applicable to X and $\{X_i\}_{i\in I}$.

Definition 2.13. For two continuous maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, the *fibre product* is defined as

$$X_1 \times_Y X_2 := \{(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2 \mid f_1(\mathbf{x}_1) = f_2(\mathbf{x}_2)\}.$$

Theorem 2.14 [10, Theorem 1]. *Let $f : X \to Y$ be a continuous closed surjective o-minimal map. Then there is a spectral sequence $E^r_{p,q}$ converging to $H_*(Y)$ with $E^1_{p,q} = H_q(W_p)$, where*

$$W_p := \underbrace{X \times_Y \ldots \times_Y X}_{p+1 \text{ times}}.$$

Corollary 2.15. *For $f : X \to Y$ as in Theorem 2.14 and for any $k \geqslant 0$, we have*

$$\mathrm{b}_k(Y) \leqslant \sum_{p+q=k} \mathrm{b}_q(W_p),$$

where $\mathrm{b}_k := \mathrm{rank} H_k$ is the kth Betti number.

3. *Simplicial construction*

Since G and S are definable, they are triangulable [6, Theorem 4.4], that is, there exists a finite simplicial complex $R = \{\Delta_j\}$ and a definable homeomorphism $\Phi : |R| \to G$, where $|R|$ is the geometric realization of R, such that S is a union of images under Φ of some simplices of R.

By a *simplex* we always mean an *open simplex*. If Δ is a simplex, then $\overline{\Delta}$ denotes its closure. In what follows we will ignore the distinction between simplices of $|R|$ and their images in G.

Definition 3.1. For a simplex Δ of S, its *subsimplex* is a simplex $\Delta' \neq \Delta$ such that $\Delta' \subset \overline{\Delta}$. A *k-flag* of simplices of R is a sequence $\Delta_{i_0}, \dots, \Delta_{i_k}$ such that Δ_{i_ν} is a subsimplex of $\Delta_{i_{\nu-1}}$ for $\nu = 1, \dots, k$.

Definition 3.2. The set S is *marked* if, for every pair (Δ', Δ) of simplices of S such that Δ' is a subsimplex of Δ, the simplex Δ' is classified as either a *hard* or *soft* subsimplex of Δ. If Δ' is not in S, then it is always soft.

In what follows we assume that S is marked.

Let $\widehat{R}$ be the barycentric subdivision of R. Then each vertex v_j of $\widehat{R}$ is the centre of a simplex Δ_j of R. Let $B = B(j_0, \dots, j_k)$ be a k-simplex of $\widehat{R}$ having vertices $v_{j_0}, \dots, v_{j_k}$. Assume that the vertices of B are ordered so that $\dim \Delta_{j_0} > \dots > \dim \Delta_{j_k}$. Then B corresponds to a k-flag $\Delta_{j_0}, \dots, \Delta_{j_k}$ of simplices of R. Let $\widehat{S}$ be the set of simplices of $\widehat{R}$ that belong to S. Then S is the union of all simplices of $\widehat{S}$.

Definition 3.3. The *core* $C(B)$ of a simplex $B = B(j_0, \dots, j_k)$ of $\widehat{S}$ is the maximal subset $\{j_0, \dots, j_p\}$ of the set $\{j_0, \dots, j_k\}$ such that Δ_{j_ν} is a *hard* subsimplex of Δ_{j_μ} for all $\mu < \nu \leqslant p$. Note that j_0 is always in $C(B)$, and, in particular, $C(B) \neq \emptyset$. Assume that, for a simplex B not in $\widehat{S}$, the core $C(B)$ is empty.

Lemma 3.4. *Let $B = B(i_0, \dots, i_k)$ be a simplex in $\widehat{S}$ and $K = K(j_0, \dots, j_\ell)$ be a simplex in $\widehat{R}$, with $B \subset \overline{K}$, that is, $I = \{i_0, \dots, i_k\} \subset J = \{j_0, \dots, j_\ell\}$. Then $I \setminus C(B) \subset J \setminus C(K)$.*

The proof of this lemma is a straightforward consequence of the definitions.

Definition 3.5. For two simplices B and B' of $\widehat{S}$, let $B' \succeq B$ if either B' is a subsimplex of B (reverse inclusion) and $C(B') \cap C(B) = \emptyset$, or $B' = B$. If $B' \succeq B$ and $B' \neq B$, then we write $B' \succ B$. Lemma 3.4 implies that $\succeq$ is a partial order on the set of all simplices of $\widehat{S}$. The *rank* $r(\widehat{S})$ of $\widehat{S}$ is the maximal length r of a chain $\Delta_0 \succ \dots \succ \Delta_r$ of simplices in $\widehat{S}$. Let B be a simplex in $\widehat{S}$. The set S_B of simplices $B' \subset \overline{B} \cap \widehat{S}$ is a poset with partial order induced from $(\widehat{S}, \succeq)$. The *rank* $r(S_B)$ *of* S_B is the maximal length of its chain.

Definition 3.6. Let simplices B and K be as in Lemma 3.4. For $0 < \delta < 1$, define

$$B(\delta) := \left\{ \sum_{i_\nu \in I} t_{i_\nu} v_{i_\nu} \in B(i_0, \dots, i_k) \;\middle|\; \sum_{i_\nu \in C(B)} t_{i_\nu} > \delta \right\}.$$

For $0 < \varepsilon < 1$ and $0 < \delta < 1$, define

$$K_B(\delta, \varepsilon) := \Bigg\{ \sum_{j_\nu \in J} t_{j_\nu} v_{j_\nu} \in K(j_0, \dots, j_\ell) \;\Bigg|\; \sum_{i_\nu \in C(B)} t_{i_\nu} > \delta,$$
$$\sum_{i_\nu \in I} t_{i_\nu} > 1 - \varepsilon,\ \forall i_\nu \in I\ \forall j_\mu \in (J \setminus I)\ (t_{i_\nu} > t_{j_\mu}) \Bigg\}.$$

Definition 3.7. Let B be a simplex in $\widehat{S}$. Fix some $m \geqslant 0$ and a sequence $\varepsilon_0, \delta_0, \varepsilon_1, \delta_1, \ldots, \varepsilon_m, \delta_m$. Define V_B as the union of sets $K_{B'}(\delta_i, \varepsilon_i)$ over all simplices $B' \in S_B$, all simplices K of $\widehat{R}$ such that $B \subset \overline{K}$, and for $i = 0, \ldots, m$. Define V as the union of the sets V_B over all simplices B of $\widehat{S}$.

4. *Topological relations between V and S*

Lemma 4.1. *Let $B = B(i_0, \ldots, i_k)$ be a simplex in $\widehat{S}$ and $K = K(j_0, \ldots, j_\ell)$ be a all simplex in $\widehat{R}$, with $B \subset \overline{K}$. Then*

$$K_B(\delta, \varepsilon) \cap K_B(\delta', \varepsilon') = K_B(\max\{\delta, \delta'\}, \min\{\varepsilon, \varepsilon'\})$$

for all $0 < \delta, \varepsilon, \delta', \varepsilon' < 1$.

The proof of this lemma is a straightforward consequence of the definitions.

Lemma 4.2. *For any two simplices B and B' of $\widehat{S}$, a simplex K of $\widehat{R}$ such that B and B' are subsimplices of K, and all $0 < \delta, \varepsilon, \delta', \varepsilon' < 1$, the following hold:*

(i) *if $K_B(\delta, \varepsilon) \cap K_{B'}(\delta', \varepsilon') \neq \emptyset$, then either $B \subset \overline{B'}$ or $B' \subset \overline{B}$;*

(ii) *$K_B(\delta, \varepsilon) \cap K_{B'}(\delta', \varepsilon')$ is convex.*

The proof of this lemma is a straightforward consequence of the definitions.

Lemma 4.3. *Let K be a simplex of $\widehat{R}$ and let $B_0, \ldots, B_k$ be a flag of simplices of $\widehat{S}$, with $B_0 \subset \overline{K}$. Then for (1.1) and a sequence $i_0, j_0, \ldots, i_k, j_k$ of integers in $\{0, 1, \ldots, m\}$, the intersection*

$$Z_K(i_0, j_0, \ldots, i_k, j_k) := K_{B_0}(\delta_{i_0}, \varepsilon_{j_0}) \cap \ldots \cap K_{B_k}(\delta_{i_k}, \varepsilon_{j_k})$$

is non-empty if and only if $B_\mu \succ B_\nu$ implies that $j_\mu > i_\nu$ for any $\mu, \nu \in \{0, 1, \ldots, k\}$.

Proof. The necessity of the condition is straightforward. To show that it is sufficient we will construct a point $v := \sum t_j v_j$, where the sum is taken over all vertices v_j of K, such that $v \in Z_K(i_0, j_0, \ldots, i_k, j_k)$. This will be done in three steps.

(a) Define ℓ_ν as the last index in $C(B_\nu)$ (that is, v_{ℓ_ν} is the centre of the smallest simplex Δ_j of R such that $j \in C(B_\nu)$); set $t_{\ell_\nu} := \delta_{i_\nu}$. If ℓ_ν is the same index for several ν, then set t_{ℓ_ν} to be the maximum of the corresponding δ_{i_ν}.

(b) Fix a sequence $\gamma_0, \ldots, \gamma_{k+1}$ such that $0 < \gamma_0 < \ldots < \gamma_{k+1} \ll \varepsilon_0$. For a vertex v_j of $B_{\nu-1}$ that is not one of the v_{ℓ_μ} and not a vertex of B_ν, set $t_j := \gamma_\nu + \max \delta_{i_\mu}$, where the maximum is taken over all μ such that $B_\nu \succ B_\mu$ (or equals 0 if there is no such μ). For any vertex v_j of K that does not belong to B_0, set $t_j := \gamma_0$. For a vertex v_j of B_k that is not one of the v_{ℓ_ν}, set $t_j := \gamma_{k+1} + \max \delta_{i_\mu}$, where the maximum is taken over $\mu = 0, \ldots, k$.

(c) For the last vertex v_ω of B_k, set $t_\omega := 1 - \sum_{v_j} t_j$, where the sum is taken over all vertices v_j of K other than v_ω. If $\omega = \ell_k$, then this overrides the setting in (a). If $\omega \neq \ell_k$, then this overrides the setting in (b).

It is easy to check that $v \in Z_K(i_0, j_0, \ldots, i_k, j_k)$. $\square$

Lemma 4.4. *Let $B_0, \ldots, B_k$ be a flag of simplices of $\widehat{S}$, and $i_0, j_0, \ldots, i_k, j_k$ be a sequence of integers in $\{0, 1, \ldots, m\}$. For (1.1), if $B_\mu \succ B_\nu$ implies that $j_\mu > i_\nu$ for any $\mu, \nu \in \{0, 1, \ldots, k\}$,* *then the set*

$$Z(i_0, j_0, \dots, i_k, j_k) := \bigcup_K Z_K(i_0, j_0, \dots, i_k, j_k),$$

where the union is taken over all simplices K of $\widehat{R}$ with $B_0 \subset \overline{K}$, is an open contractible subset of G. Otherwise, $Z(i_0, j_0, \dots, i_k, j_k) = \emptyset$.

Proof. By Lemma 4.3, $Z(i_0, j_0, \dots, i_k, j_k) \neq \emptyset$ if and only if $B_\mu \succ B_\nu$ implies that $j_\mu > i_\nu$ for any $\mu, \nu \in \{0, 1, \dots, k\}$. So, suppose that $Z(i_0, j_0, \dots, i_k, j_k) \neq \emptyset$, and consider two simplices K and K' such that $B_0 \subset \overline{K'} \subset \overline{K}$. Then the intersection of the closure of the complement in K of $Z_K(i_0, j_0, \dots, i_k, j_k)$ with K' coincides with the complement in K' of $Z_{K'}(i_0, j_0, \dots, i_k, j_k)$. Hence, $Z_{K'}(i_0, j_0, \dots, i_k, j_k) \cup Z_K(i_0, j_0, \dots, i_k, j_k)$ is open in $\overline{K}$. It follows that $Z(i_0, j_0, \dots, i_k, j_k)$ is open in G and has a closed covering by convex sets $\overline{Z_K(i_0, j_0, \dots, i_k, j_k)} \cap Z(i_0, j_0, \dots, i_k, j_k)$ over all simplices K of $\widehat{R}$. This covering has the same nerve as the star of B_0 in the complex $\widehat{R}$, and this star is contractible. An intersection of any number of elements of the covering of $Z(i_0, j_0, \dots, i_k, j_k)$ is convex, and therefore contractible. By the nerve theorem (Theorem 2.10(ii)), both $Z(i_0, j_0, \dots, i_k, j_k)$ and the star are homotopy equivalent to the geometric realization of the nerve, and hence also homotopy equivalent to one another. It follows that $Z(i_0, j_0, \dots, i_k, j_k)$ is contractible. $\square$

LEMMA 4.5. *For (1.1) and for each simplex B in $\widehat{S}$ and for every $m \geqslant 1$, the set V_B (see Definition 3.7) is open in G and $(m-1)$-connected.*

Proof. For every simplex $B' \in S_B$, consider the set $U_{B',i} := \bigcup_K K_{B'}(\delta_i, \varepsilon_i)$, where the union is taken over all simplices K of $\widehat{R}$ with $B \subset \overline{K}$. Obviously, the family $\{U_{B',i} \mid B' \in S_B, 0 \leqslant i \leqslant m\}$ is an open covering of V_B. Let M_B denote the nerve of this covering. From Lemmas 4.1, and 4.4, M_B is the simplicial complex whose k-simplices can be identified with all sequences of the kind $((p_0, i_0), \dots, (p_k, i_k))$, where p_ν are indices of the simplices $B'_{p_\nu} \in S_B$, such that the following hold:

(a) $B'_{p_\nu} \subset \overline{B'_{p_{\nu-1}}}$;
(b) $0 \leqslant i_\nu \leqslant m$;
(c) if $B'_{p_\mu} \succeq B'_{p_\nu}$ and $\mu > \nu$, then $i_\mu > i_\nu$.

By Lemma 4.4, any non-empty intersection of sets $U_{B',i}$ is contractible. Therefore, by the nerve theorem (Theorem 2.10(ii)), V_B is homotopy equivalent to M_B, and in order to prove that V_B is $(m-1)$-connected it is sufficient to show that M_B is an $(m-1)$-connected simplicial complex. This follows from Proposition 4.6 below. $\square$

Let $\succeq$ be a poset on $\{0, \dots, N\}$ such that, if $p \succeq q$ and $p \neq q$, then $p > q$. For each $p \in \{0, \dots, N\}$, let $r(p)$ be the maximal length of a poset chain with the maximal element p (that is, the rank of the order ideal generated by p). Let $m_0, \dots, m_N$ be non-negative integers. Let $M(m_0, \dots, m_N)$ be the simplicial complex containing all k-simplices $((p_0, i_0), \dots, (p_k, i_k))$ such that the following hold:

(a) $p_\nu \in \{0, \dots, N\}$, where $p_0 \leqslant \ldots \leqslant p_k$;
(b) $i_\nu \in \{0, \dots, m_\nu\}$;
(c) if $p_\mu \succeq p_\nu$ and $\mu > \nu$, then $i_\mu > i_\nu$.

Let $m := \min\{m_1, \dots, m_N\}$.

An example of the complex $M(2, 2)$ with $1 \succ 0$ is shown in Figure 2.

PROPOSITION 4.6. *The simplicial complex $M(m_0, \dots, m_N)$ is $(m-1)$-connected.*

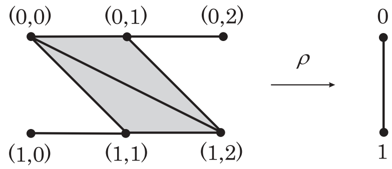

FIGURE 2. *The complex $M(2,2)$ with $1 \succ 0$.*

Proof. Let Δ^N be the N-simplex and $\overline{\Delta^N(m)}$ be the m-dimensional skeleton of its closure. There is a natural simplicial map

$$\begin{aligned} \rho : M(m_0, \dots, m_N) &\longrightarrow \overline{\Delta^N}, \\ (p, i) &\longmapsto p. \end{aligned}$$

It is easy to see that $\overline{\Delta^N(m)} \subset \rho(M(m_0, \dots, m_N))$, and hence $\rho(M(m_0, \dots, m_N))$ is $(m-1)$-connected.

Consider any face Δ^L of Δ^N, that $L \leqslant N$, that has a non-empty pre-image under ρ. Without loss of generality, assume that its vertices are $0, \dots, L$. Let $M(m_0, \dots, m_L)$ be the simplicial complex defined over the poset on $\{0, \dots, L\}$ induced by $\succeq$. We prove inductively on L that, for any point $\mathbf{x} \in \Delta^L$, the fibre $\rho^{-1}(\mathbf{x})$ is contractible. The proposition then follows from the Vietoris–Begle theorem (Corollary 2.6(ii)). The base of induction, for $L = 0$, is obvious. Assume that the statement is true for $L - 1$. For any simplex $K = ((p_0, i_0), \dots, (p_k, i_k))$ of $M(m_0, \dots, m_L)$ that projects surjectively onto Δ^L, if $p_\nu = L$ then $i_\nu \geqslant r(L)$. Let $s = i_\ell$ be the minimal of these i_ν in K, so that $p_\nu < L$ for $\nu < \ell$, while $p_\ell = L$. Then $((p_0, i_0), \dots, (p_{\ell-1}, i_{\ell-1}))$ is a simplex of the simplicial complex $M(s) := M(m'_0, \dots, m'_{L-1})$, defined over the poset on $\{0, \dots, L-1\}$ induced by $\succeq$, where $m'_p := \min\{m_p, r(p) + s - r(L)\}$ if $L \succeq p$, and $m'_p := m_p$ if L is incomparable with p. It follows that K is a simplex of the join of $M(s)$ and $\overline{\Delta^{m_L - s}}$, where $\Delta^{m_L - s}$ is the simplex with vertices $s, \dots, m_L$. Since the complex $M(s)$ is contractible by the induction hypothesis, its join with $\overline{\Delta^{m_L - s}}$ has a contractible fibre over any $\mathbf{x} \in \Delta^L$. The fibre over $\mathbf{x}$ of $M(m_0, \dots, m_L)$ is the union of these contractible fibres for $s = r(N), \dots, m_L$. The intersection of any number of these fibres is non-empty and contractible, being a fibre of the join of $M_{\min}$ and $\overline{\Delta^{m_L - s_{\max}}}$. By the nerve theorem (Theorem 2.10(ii)), their union is homotopy equivalent to its nerve, a simplex, and thus is contractible. □

COROLLARY 4.7. *In the definition of the simplicial complex $M(m_0, \dots, m_N)$ assume additionally that $m_j \geqslant r(j)$ for every $j = 0, \dots, N$. Then $M(m_0, \dots, m_N)$ is contractible.*

Proof. The condition $m_j \geqslant r(j)$ guarantees that the map ρ is surjective, and hence $\rho(M(m_0, \dots, m_N))$ is contractible. □

THEOREM 4.8. *For (1.1), there are homomorphisms $\chi_k : H_k(V) \to H_k(S)$ and $\tau_k : \pi_k(V) \to \pi_k(S)$ such that χ_k and τ_k are isomorphisms for every $k \leqslant m-1$, and χ_m and τ_m are epimorphisms. Moreover, if $m \geqslant \dim(S)$, then $V \simeq S$.*

Proof. By Lemma 4.2(i), for any three simplices B_0, B_1, and B_2 in $\widehat{S}$, the equality $\overline{B}_0 = \overline{B}_1 \cap \overline{B}_2$ is equivalent to $V_{B_0} = V_{B_1} \cap V_{B_2}$. Hence, a non-empty intersection of any number of sets V_B is a set of the same type, and therefore is $(m-1)$-connected. Moreover, there is an isomorphism $\xi : |\mathcal{N}_V| \to |\mathcal{N}_{\widehat{S}}|$ between the geometric realization of the nerve $\mathcal{N}_{\widehat{S}}$ of the covering of $\widehat{S}$ by its simplices and the geometric realization of the nerve $\mathcal{N}_V$ of the open covering of V by sets V_B.

Since intersections of any number of elements of the covering of $\widehat{S}$ (that is, simplices) are contractible if non-empty, the nerve theorem (Theorem 2.10(ii) and Remark 2.12) implies that $\widehat{S} \simeq |\mathcal{N}_{\widehat{S}}|$, that is, there is a continuous map $\psi_{\widehat{S}} : \widehat{S} \to |\mathcal{N}_{\widehat{S}}|$ that induces isomorphisms of homotopy groups $\psi_{\widehat{S}\#} : \pi_k(\widehat{S}) \to \pi_k(|\mathcal{N}_{\widehat{S}}|)$ for all integers $k \geqslant 0$.

On the other hand, by the nerve theorem (Theorem 2.10(i)), there is a continuous map $\psi_V : V \to |\mathcal{N}_V|$ inducing isomorphisms of homotopy groups $\psi_{V\#} : \pi_k(V) \to \pi_k(|\mathcal{N}_V|)$ for every $k \leqslant m-1$ and an epimorphism $\psi_{V\#} : \pi_m(V) \to \pi_m(|\mathcal{N}_V|)$. Here, we let τ_k be

$$\psi_{\widehat{S}\#}^{-1} \circ \xi \circ \psi_{V\#} : \pi_k(V) \longrightarrow \pi_k(\widehat{S}).$$

By the Whitehead theorem on homotopy and homology (Theorem 2.2), $\psi_{\widehat{S}}$ induces isomorphisms of homology groups $\psi_{\widehat{S}*} : H_k(\widehat{S}) \to H_k(|\mathcal{N}_{\widehat{S}}|)$ for all $k \geqslant 0$, while ψ_V induces isomorphisms of homology groups $\psi_{V*} : H_k(V) \to H_k(|\mathcal{N}_V|)$ for every $k \leqslant m-1$, and an epimorphism $\psi_{V*} : H_m(V) \to H_m(|\mathcal{N}_V|)$. We let χ_k be

$$\psi_{\widehat{S}*}^{-1} \circ \xi \circ \psi_{V*} : H_k(V) \longrightarrow H_k(\widehat{S}).$$

If $m \geqslant \dim(S)$ then, by Corollary 4.7, a non-empty intersection of any number of sets V_B is contractible. Then, according to the nerve theorem, the sets V and $\widehat{S}$ are homotopy equivalent to geometric realizations of the respective nerves and therefore $V \simeq S$. □

5. *Proof of Theorem* 1.10

We now need to re-define the simplicial complex R so that it will satisfy additional properties. Recall that definable functions are triangulable [**6**, Theorem 4.5]. Consider a finite simplicial complex R' such that R' is a triangulation of the projection

$$\rho : G \times [0,1] \longrightarrow [0,1],$$

and R' is compatible with

$$\bigcup_{\delta \in (0,1)} (S_\delta, \delta) \subset G \times [0,1].$$

Define R as the triangulation induced by R' on the fibre $\rho^{-1}(0)$.

Definition 5.1. Along with the sequence $\varepsilon_0, \dots, \delta_m$, consider another sequence $\varepsilon_0', \delta_0', \varepsilon_1', \delta_1', \dots, \varepsilon_m', \delta_m'$. Let T' be the set defined as in Definition 1.8 replacing all δ_i and ε_i by δ_i' and ε_i', respectively. Let V' be the set defined as in Definition 3.7 replacing all δ_i and ε_i by δ_i' and ε_i', respectively.

5.1. *Definable case*

In the definable case we specify the hard–soft relation for the set V' as follows. For any pair (Δ_1, Δ_2) of S such that Δ_1 is a subsimplex of Δ_2, we assume that Δ_1 is *soft* in Δ_2.

Definition 5.2. Let B and K be as in Lemma 3.4. For $0 < \varepsilon < 1$, define

$$K_B(\varepsilon) := \left\{ \sum_{j_\nu \in J} t_{j_\nu} v_{j_\nu} \in K(j_0, \dots, j_\ell) \,\middle|\, \sum_{i_\nu \in I} t_{i_\nu} > 1 - \varepsilon,\ \forall i_\nu \in I\ \forall j_\mu \in (J \setminus I)\ (t_{i_\nu} > t_{j_\mu}) \right\}.$$

Introduce a new parameter ε'' and define V'' as the union of $K_B(\varepsilon'')$ over all simplices B of $\widehat{S}$, and all simplices K of $\widehat{R}$ such that $B \subset \overline{K}$.

In each of the following Lemmas 5.3–5.6, the statement holds for

$$0 < \varepsilon_0' \ll \ldots \ll \varepsilon_i' \ll \varepsilon_i \ll \delta_i \ll \delta_i' \ll \ldots \ll \delta_m' \ll \varepsilon'' \quad (i = 0, \ldots, m).$$

Lemma 5.3. *We have $S \simeq V''$.*

Proof. Let B be a simplex of $\widehat{S}$ and let $U_B := \bigcup K_B(\varepsilon'')$, where the union is taken over all simplices K of $\widehat{R}$ such that $B \subset \overline{K}$. Then the family of all sets U_B forms an open covering of V'' whose nerve we denote by $\mathcal{N}_{V''}$. Each U_B is contractible, since B is a deformation retract of U_B. Any intersection $U := U_{B_0} \cap \ldots \cap U_{B_k}$ is non-empty if and only if, after the suitable reordering, the sequence $B_{i_1}, \ldots, B_{i_k}$ is a k-flag of simplices. If $U \neq \emptyset$, then B_{i_k} is its deformation retract, and hence U is contractible. By the nerve theorem (Theorem 2.10(ii)), we have $V'' \simeq |\mathcal{N}_{V''}|$. On the other hand, the simplices of $\widehat{S}$ form a covering of S with nerve $\mathcal{N}_S$ (in the sense of Remark 2.12), and therefore $S \simeq |\mathcal{N}_S|$ by the nerve theorem. Then $S \simeq V''$, since the nerves $\mathcal{N}_{V''}$ and $\mathcal{N}_S$ are isomorphic. □

Lemma 5.4. *We have $V' \subset T \subset V''$.*

Proof. Let $V_{\delta,\varepsilon}$ be the union of sets $K_B(\delta, \varepsilon)$ over all simplices K of $\widehat{R}$ and simplices B of $\widehat{S}$ such that $B \subset \overline{K}$. We first show that $V_{\delta',\varepsilon'} \subset S_{\delta,\varepsilon}$, which immediately implies that $V' \subset T$. Fix δ', and let $\mathbf{x}_{\varepsilon'} \in V_{\delta',\varepsilon'}$ be a definable curve. Then $\mathbf{x}_0 := \lim_{\varepsilon' \searrow 0} \mathbf{x}_{\varepsilon'} \in \overline{B(\delta')}$, where B is a simplex in $\widehat{S}$ (this follows from Definition 3.6). Let Δ be the simplex in S containing $\mathbf{x}_0$. Since every subsimplex of Δ is soft in Δ, then $\mathbf{x}_0 \in S_\delta$ for $\delta \ll \delta'$. Also, an open neighbourhood of $\mathbf{x}_0$ in G, whose size is independent of ε', is contained in $S_{\delta,\varepsilon}$ for $\varepsilon \ll \delta \ll \delta'$. Hence $\mathbf{x}_{\varepsilon'} \in S_{\delta,\varepsilon}$ for $\varepsilon' \ll \varepsilon \ll \delta \ll \delta'$.

Next, we show that $S_{\delta,\varepsilon} \subset V''$, and therefore $T \subset V''$. Fix δ, and let $\mathbf{x}_\varepsilon \in S_{\delta,\varepsilon}$ be a definable curve. Then $\mathbf{x}_0 := \lim_{\varepsilon \searrow 0} \mathbf{x}_\varepsilon \in S_\delta$. Hence $\mathbf{x}_0$ belongs to a simplex B of $\widehat{S}$. According to Definition 5.2, an open neighbourhood of $\mathbf{x}_0$ of radius larger than ε is contained in $K_B(\varepsilon'')$ for any simplex K of $\widehat{R}$ such that $B \subset \overline{K}$. In particular, $\mathbf{x}_\varepsilon \in K_B(\varepsilon'')$, and therefore $\mathbf{x}_\varepsilon \in V''$. □

Lemma 5.5. *The inclusion map $\iota : V' \hookrightarrow V''$ induces isomorphisms of homotopy groups $\iota_{k\#} : \pi_k(V') \to \pi_k(V'')$ for every $k \leqslant m-1$, and an epimorphism $\iota_{m\#}$.*

Proof. Recall that V' admits an open covering by sets of the kind $V_B' := V_B$ (Definition 3.7) over all simplices B in $\widehat{S}$ such that every non-empty intersection of the sets V_B' is $(m-1)$-connected (Lemma 4.5). Similarly, the set V'' has an open covering by the sets V_B'', where V_B'' is the union of the sets $K_{B'}(\varepsilon'')$ over all simplices $B' \in S_B$, and simplices K of $\widehat{R}$ such that $B \subset \overline{K}$. Every non-empty intersection of the sets V_B'' is contractible (cf. the proof of Lemma 5.3).

The inclusion relation $V_B' \subset V_B''$ implies that these two coverings have the same nerve $\mathcal{N}$, up to isomorphism. Let $\mathcal{N}^{(m)}$ be the m-skeleton of $\mathcal{N}$. Following the proof of Theorem 6 in [**5**], we now describe a *carrier* (see Definition 2.7) C' assigning certain intersections of the sets V_B' to simplices σ of the barycentric subdivision of $\mathcal{N}^{(m)}$.

Let Q be the face poset of $\mathcal{N}^{(m)}$ (that is, simplices ordered by a subsimplex relation) and $\Delta(Q)$ be its order complex (see Definition 2.3). Then $\Delta(Q)$ is homeomorphic to $|\mathcal{N}^{(m)}|$, being its barycentric subdivision. For σ in $\Delta(Q)$, let

$$C'(\sigma) = \bigcap_{V'_B \in \min \sigma} V'_B.$$

In a similar way a carrier C'' from simplices in the barycentric subdivision of $\mathcal{N}$ to intersections of sets V''_B is defined.

According to the carrier lemma (ii) (Theorem 2.8(ii)), there exist continuous maps $g' : \mathcal{N}^{(m)} \to V'$ and $g'' : \mathcal{N} \to V''$ such that g' is carried by C' and g'' is carried by C''. On the other hand, g' is also carried by C'' because $V'_B \subset V''_B$ implies that $g'(\sigma) \subset C'(\sigma) \subset C''(\sigma)$. Since all non-empty intersections of V''_B are contractible, and in particular, m-connected, the carrier lemma (i) implies that

$$\iota \circ g' \sim g''|_{\mathcal{N}^{(m)}}. \tag{5.1}$$

By the nerve theorem (Theorem 2.10(ii); details in [5]), g'' is a homotopy equivalence. Passing to homomorphisms of homotopy groups, we have that $g''_{\#k} : \pi_k(\mathcal{N}) \to \pi_k(V'')$ is an isomorphism for all k. Hence, by (5.1), $\iota_{\#k}$ is an epimorphism for all $k \leqslant m$. According to the nerve theorem, $g'_{\#k} : \pi_k(\mathcal{N}^{(m)}) \to \pi_k(V'')$ is an isomorphism for $k \leqslant m-1$, and thus $\iota_{\#k}$ is also a monomorphism for $k \leqslant m-1$. □

Lemma 5.6. *For every $k \leqslant m$, there are epimorphisms $\zeta_k : \pi_k(T) \to \pi_k(V'')$ and $\eta_k : H_k(T) \to H_k(V'')$.*

Proof. By Lemmas 5.4 and 5.5, we have

$$V' \overset{p}{\hookrightarrow} T \overset{q}{\hookrightarrow} V'',$$

where $\hookrightarrow$ are the inclusion maps and $q \circ p$ induces isomorphisms $(q \circ p)_\# = q_\# \circ p_\#$ of homotopy groups $\pi_k(V') \cong \pi_k(V'')$ for every $k \leqslant m-1$, and an epimorphism $\pi_m(V') \to \pi_m(V'')$. Then $\zeta_k := q_\#$ is an epimorphism for every $k \leqslant m$.

By the Whitehead theorem on homotopy and homology (Theorem 2.2), $q \circ p$ also induces isomorphisms $(q \circ p)_* = q_* \circ p_*$ of homology groups $H_k(V') \cong H_k(V'')$ for every $k \leqslant m-1$, and an epimorphism $H_k(V') \to H_k(V'')$. Hence, $\eta_k := q_*$ is an epimorphism for every $k \leqslant m$. □

Theorem 1.10(i) immediately follows from Lemmas 5.3 and 5.6.

5.2. *Separability and constructible case*

Definition 5.7. For the simplicial complex R and the family $\{S_\delta\}_{\delta>0}$, we call the pair $(R, \{S_\delta\}_{\delta>0})$ *separable* if, for any pair (Δ_1, Δ_2) of simplices of S such that Δ_1 is a subsimplex of Δ_2, the equality $\overline{\Delta_2 \cap S_\delta} \cap \Delta_1 = \emptyset$ is equivalent to the inclusion $\Delta_1 \subset \overline{\Delta_2 \setminus S_\delta}$ for all sufficiently small $\delta > 0$.

Recall that, in the constructible case, we assume that S is defined by a Boolean combination of equations and inequalities with continuous definable functions, and the set S_δ is defined using sign sets of these functions (see Section 1).

Lemma 5.8. *In the constructible case, $(R, \{S_\delta\}_{\delta>0})$ is separable.*

Proof. Observe that R is compatible with the sign set decomposition of S.

Consider a pair (Δ_1, Δ_2) of simplices of S such that Δ_1 is a subsimplex of Δ_2. If both Δ_1 and Δ_2 lie in the same sign set, then $\overline{\Delta_2 \cap S_\delta} \cap \Delta_1 = \overline{\Delta_1 \cap S_\delta} \neq \emptyset$ and $\Delta_1 \not\subset \overline{\Delta_2 \setminus S_\delta}$.

If Δ_1 and Δ_2 lie in two different sign sets, then there is a function h in the Boolean combination defining S such that $h(\mathbf{x}) = 0$ for every point $\mathbf{x} \in \Delta_1$, while $h(\mathbf{y})$ satisfies a strict inequality, say $h(\mathbf{y}) > 0$, for every point $\mathbf{y} \in \Delta_2$. Then $\overline{\Delta_2 \cap S_\delta} \subset \overline{\Delta_2 \cap \{h \geqslant \delta\}}$ and $\overline{\Delta_2 \cap \{h \geqslant \delta\}} \cap \{h = 0\} = \emptyset$. Hence $\overline{\Delta_2 \cap S_\delta} \cap \Delta_1 = \emptyset$. On the other hand, $\overline{\Delta_2 \setminus S_\delta} \supset \overline{\Delta_2 \cap \{h < \delta\}} \supset \overline{\Delta_2 \cap \{h = 0\}} \supset \Delta_1$. □

Now we return to the general definable case and assume, in the rest of this section, that $(R, \{S_\delta\})$ is separable. For any pair (Δ_1, Δ_2) of simplices of S such that Δ_1 is a subsimplex of Δ_2, we assume that Δ_1 is *soft* in Δ_2 if $\overline{\Delta_2 \cap S_\delta} \cap \Delta_1 = \emptyset$ (equivalently, $\Delta_1 \subset \overline{\Delta_2 \setminus S_\delta}$) for all sufficiently small $\delta > 0$. Otherwise, Δ_1 is *hard* in Δ_2.

LEMMA 5.9. *If Δ_1 is hard in Δ_2, then, for every $\mathbf{x} \in \Delta_1$, there is a neighbourhood $U_{\mathbf{x}}$ of $\mathbf{x}$ in $\overline{\Delta_2}$ such that, for all sufficiently small $\delta \in (0, 1)$, we have $U_{\mathbf{x}} \subset \overline{\Delta_2 \cap S_\delta}$.*

Proof. Suppose that, contrary to the claim, for some $\mathbf{x} \in \Delta_1$, we have $U_{\mathbf{x}} \setminus \overline{\Delta_2 \cap S_\delta} \neq \emptyset$ for any neighbourhood $U_{\mathbf{x}}$ of $\mathbf{x}$ in Δ_1, for arbitrarily small $\delta > 0$.

Since the set S_δ grows (with respect to inclusion) as $\delta \searrow 0$, and Δ_1 is hard in Δ_2, the intersection $\overline{\Delta_2 \cap S_\delta} \cap \Delta_1$ is non-empty and also grows. If, for any neighbourhood $W_{\mathbf{x}}$ of $\mathbf{x}$ in Δ_1, we have $W_{\mathbf{x}} \not\subset \overline{\Delta_2 \cap S_\delta} \cap \Delta_1$ for arbitrarily small $\delta > 0$, then the limits of both $\overline{\Delta_2 \cap S_\delta} \cap \Delta_1$ and its complement in Δ_1, as $\delta \searrow 0$, have non-empty intersections with Δ_1. This contradicts the assumption that Δ_1 is a simplex in the complex R compatible with R', and thus there is a neighbourhood $W_{\mathbf{x}}$ in Δ_1 such that $W_{\mathbf{x}} \subset \overline{\Delta_2 \cap S_\delta} \cap \Delta_1$ for sufficiently small $\delta > 0$. It follows that $U_{\mathbf{x}} \setminus \overline{\Delta_2 \cap S_\delta} \subset \Delta_2$. Since $\mathbf{x} \in \overline{\Delta_2 \setminus S_\delta}$, and again using the compatibility of the complex R with R', we conclude that $\Delta_1 \subset \overline{\Delta_2 \setminus S_\delta}$, that is, Δ_1 is soft in Δ_2, which is a contradiction. □

In each of the following Lemmas 5.10 and 5.11 and Theorem 5.12 the statement holds for

$$0 < \varepsilon'_0 \ll \ldots \ll \varepsilon'_i \ll \varepsilon_i \ll \delta_i \ll \delta'_i \ll \ldots \ll \delta'_m \ll 1 \quad (i = 0, \ldots, m). \tag{5.2}$$

LEMMA 5.10. *We have $T' \subset V$ and $V' \subset T$.*

Proof. We show first that $S_{\delta', \varepsilon'} \subset V_{\delta, \varepsilon}$ for $\varepsilon' \ll \varepsilon \ll \delta \ll \delta'$, where $V_{\delta, \varepsilon}$ is the union of $K_B(\delta, \varepsilon)$ over all simplices K of $\widehat{R}$ and simplices B of $\widehat{S}$ such that $B \subset \overline{K}$.

Let us fix δ', and let $\mathbf{x}_{\varepsilon'} \in S_{\delta', \varepsilon'}$ be any definable curve. It is enough to show that $\mathbf{x}_{\varepsilon'} \in V_{\delta, \varepsilon}$ for $\varepsilon' \ll \varepsilon \ll \delta \ll \delta'$. Clearly, $\mathbf{x}_0 =: \lim_{\varepsilon' \searrow 0} \mathbf{x}_{\varepsilon'}$ belongs to $S_{\delta'}$. Hence $\mathbf{x}_0$ belongs to a simplex $B = B(j_0, \ldots, j_\ell)$ of $\widehat{S}$. Suppose that $\mathbf{x}_0 \notin B(\delta)$. Let $\mathbf{x}_{0,\delta} \in B \setminus B(\delta)$ be a definable curve. Then $\mathbf{x}_{0,0} =: \lim_{\delta \searrow 0} \mathbf{x}_{0,\delta}$ belongs to a subsimplex $B' = B(i_0, \ldots, i_k)$ of B. It follows that $\mathbf{x}_{0,0} \in \overline{\Delta_{j_0} \cap S_{\delta'}} \cap \Delta_{i_0}$, and therefore Δ_{i_0} is *hard* in Δ_{j_0}. On the other hand, by the definition of $B(\delta)$ (Definition 3.6), Δ_{i_0} is *soft* in Δ_{j_0}. This contradiction shows that $\mathbf{x}_0 \in B(\delta)$.

For $\varepsilon' \ll \varepsilon$, the distance from $\mathbf{x}_{\varepsilon'}$ to $\mathbf{x}_0 \in S_{\delta'} \cap B$ is much smaller than ε. From Definition 3.6, for $\varepsilon \ll \delta \ll \delta'$, the union of $K_B(\delta, \varepsilon)$ over all simplices K of $\widehat{R}$ such that $B \subset \overline{K}$ contains an open in G neighbourhood of $\mathbf{x}_0 \in B$ whose size is independent of ε'. Hence $\mathbf{x}_{\varepsilon'} \in V_{\delta, \varepsilon}$ for $\varepsilon' \ll \varepsilon \ll \delta \ll \delta'$.

Next, we want to show that $V_{\delta', \varepsilon'} \subset S_{\delta, \varepsilon}$. As before, fix δ'. Let $\mathbf{x}_{\varepsilon'} \in V_{\delta', \varepsilon'}$ be a definable curve. Then $\mathbf{x}_0 := \lim_{\varepsilon' \searrow 0} \mathbf{x}_{\varepsilon'} \in B(\delta')$, where $B = B(j_0, \ldots, j_\ell)$ is a simplex in $\widehat{S}$ (this follows from Definition 3.6). Suppose that, if $\mathbf{x}_0 \notin S_\delta$, then $\mathbf{x}_0 \in B \setminus S_\delta$. Let $\mathbf{x}_{0,\delta} \in B \setminus S_\delta$ be a definable curve. Therefore $\mathbf{x}_{0,0} := \lim_{\delta \searrow 0} \mathbf{x}_{0,\delta}$ belongs to a subsimplex $B' = B(i_0, \ldots, i_k)$ of B. Then, by Lemma 5.9, Δ_{i_0} is *soft* in Δ_{j_0}, and thus $\mathbf{x}_{0,0} \notin V_{\delta', \varepsilon'}$. The same is true for $\mathbf{x}_{\varepsilon'}$ as well, namely $\mathbf{x}_{\varepsilon'} \notin V_{\delta', \varepsilon'}$ for $\varepsilon' \ll \delta \ll \delta'$. This contradiction shows that $\mathbf{x}_0 \in S_\delta$.

Since $\mathbf{x}_0 \in S$, an open neighbourhood of $\mathbf{x}_0$ in G, whose size is independent of ε', is contained in $S_{\delta,\varepsilon}$ for $\varepsilon \ll \delta \ll \delta'$. Hence $\mathbf{x}_{\varepsilon'} \in S_{\delta,\varepsilon}$ for $\varepsilon' \ll \varepsilon \ll \delta \ll \delta'$. □

Lemma 5.11. *The inclusion maps $T' \hookrightarrow T$ and $V' \hookrightarrow V$ are homotopy equivalences.*

Proof. Proofs of homotopy equivalences are similar for both the inclusions, and so we will consider only the case $T' \hookrightarrow T$.

Consider $\varepsilon_0, \delta_0, \ldots, \varepsilon_m, \delta_m$ as variables. Then $T \subset \mathbb{R}^{n+2m+2}$. From the o-minimal version of Hardt's triviality, applied to the projection $\rho : T \to \mathbb{R}^{2m+2}$ on the subspace of coordinates $\varepsilon_0, \delta_0, \ldots, \varepsilon_m, \delta_m$, there follows the existence of a partition of $\mathbb{R}^{2m+2}$ into a finite number of connected definable sets $\{A_i\}$ such that T is definably trivial over each A_i, that is, for any point $(\bar{\varepsilon}, \bar{\delta}) := (\varepsilon_0, \delta_0, \ldots, \varepsilon_m, \delta_m) \in A_i$ the pre-image $\rho^{-1}(A_i)$ is definably homeomorphic to $\rho^{-1}(\bar{\varepsilon}, \bar{\delta}) \times A_i$ by a fibre-preserving homeomorphism.

There exists an element A_{i_0} of the partition that is an open connected set in $\mathbb{R}^{2m+2}$ and contains both the points $(\bar{\varepsilon}, \bar{\delta})$ and $(\bar{\varepsilon}', \bar{\delta}')$ for (5.2). Let $\gamma : [0,1] \to A_{i_0}$ be a definable simple curve such that $\gamma(0) = (\bar{\varepsilon}, \bar{\delta})$ and $\gamma(1) = (\bar{\varepsilon}', \bar{\delta}')$. Then $\rho^{-1}(\gamma(0)) = T$, $\rho^{-1}(\gamma(1)) = T'$, and $\rho^{-1}(\gamma([0,1]))$ is definably homeomorphic to $T \times \gamma([0,1])$. Let $\Phi_{t,t'} : \rho^{-1}(\gamma(t')) \to \rho^{-1}(\gamma(t))$ for $0 \leqslant t \leqslant t' \leqslant 1$ be the homeomorphism of fibres. Replacing if necessary $(\bar{\varepsilon}, \bar{\delta})$ by a point closer to $(\bar{\varepsilon}', \bar{\delta}')$ along the curve γ, we can assume that $\rho^{-1}(\gamma(t')) \subset \rho^{-1}(\gamma(t))$ for all $0 \leqslant t \leqslant t' \leqslant 1$. Then T' is a strong deformation retract of T defined by the homotopy $F : T \times [0,1] \to T$ as follows. If $\mathbf{x} \in \rho^{-1}(\gamma(t'))$ for some $t' \leqslant t$ and $\mathbf{x} \notin \rho^{-1}(\gamma(t''))$ for any $t'' > t'$, then $F(\mathbf{x}, t) = \Phi_{t',\, t}(\mathbf{x})$. If $\mathbf{x} \in \rho^{-1}(\gamma(t'))$ with $t' > t$, then $F(\mathbf{x}, t) = \mathbf{x}$. □

Theorem 5.12. *We have $T \simeq V$.*

Proof. Consider the four sequences $(\varepsilon^{(j)}, \delta^{(j)}) := (\varepsilon_0^{(j)}, \delta_0^{j}, \ldots, \varepsilon_m^{(j)}, \delta_m^{(j)})$, where $1 \leqslant j \leqslant 4$. Let $T(\varepsilon^{(j)}, \delta^{(j)})$ and $V(\varepsilon^{(j)}, \delta^{(j)})$ be the sets defined as in Definitions 1.8 and 3.7, respectively, replacing all δ_i by $\delta_i^{(j)}$ and all ε_i by $\varepsilon_i^{(j)}$.

By Lemma 5.10, the following chain of inclusions holds:

$$T(\varepsilon^{(1)}, \delta^{(1)}) \overset{p}{\hookrightarrow} V(\varepsilon^{(2)}, \delta^{(2)}) \overset{q}{\hookrightarrow} T(\varepsilon^{(3)}, \delta^{(3)}) \overset{r}{\hookrightarrow} V(\varepsilon^{(4)}, \delta^{(4)})$$

for

$$0 < \varepsilon_0^{(j)} \ll \delta_0^{(j)} \ll \ldots \ll \varepsilon_m^{(j)} \ll \delta_m^{(j)} \ll 1,$$

where

$$\delta_{i-1}^{(j-1)} \ll \varepsilon_i^{(j-1)} \ll \varepsilon_i^{(j)} \ll \delta_i^{(j)} \ll \delta_i^{(j-1)}$$

for all $i = 1, \ldots, m$ and $j = 2, 3, 4$.

According to Lemma 5.11, $q \circ p$ and $r \circ q$ are homotopy equivalences. Passing to induced homomorphisms of homotopy groups, we have that $(q \circ p)_\# = q_\# \circ p_\#$ is an isomorphism, and hence $q_\#$ is an epimorphism. Similarly, since $(r \circ q)_\# = r_\# \circ q_\#$ is an isomorphism, $q_\#$ is a monomorphism. It follows that $q_\#$ is an isomorphism, and therefore $T \simeq V$ by the Whitehead theorem on weak homotopy equivalence (Theorem 2.1). □

Theorem 1.10(ii) immediately follows from Theorems 5.12 and 4.8.

6. *Upper bounds on Betti numbers*

The method described in this section can be applied to obtain upper bounds on Betti numbers for sets defined by Boolean formulae with functions from various classes that admit a natural measure of 'description complexity' and a suitable version of the 'Bezout theorem', most notably for semialgebraic and semi- and sub-Pfaffian sets (see, for example, [**8**]). We give detailed proofs for the semialgebraic case. The proofs can be extended to the Pfaffian case straightforwardly.

Definition 6.1. Let $f, g, h : \mathbb{N}^\ell \to \mathbb{N}$ be three functions, let $n \in \mathbb{N}$. The expression $f \leqslant \mathrm{O}(g)^n$ means that there exists $c \in \mathbb{N}$ such that $f \leqslant (cg)^n$ everywhere on $\mathbb{N}^\ell$. The expression $f \leqslant g^{\mathrm{O}(h)}$ means that there exists $c \in \mathbb{N}$ such that $f \leqslant g^{ch}$ everywhere on $\mathbb{N}^\ell$.

6.1. *Semialgebraic sets defined by quantifier-free formulae*

Consider the constructible case with $S = \{\mathbf{x} \mid \mathcal{F}(\mathbf{x})\} \subset \mathbb{R}^n$, where $\mathcal{F}$ is a Boolean combination of polynomial equations and inequalities of the kind $h(\mathbf{x}) = 0$ or $h(\mathbf{x}) > 0$, where $h \in \mathbb{R}[x_1, \dots, x_n]$. Suppose that the number of different polynomials h is s and that their degrees do not exceed d. The following upper bounds on the total Betti number $\mathrm{b}(S)$ of the set S originate from the classic works of [**12**–**14**, **17**]. Their proofs can be found in [**4**].

(a) If $\mathcal{F}$ is a conjunction of any number equations, then $\mathrm{b}(S) \leqslant d(2d-1)^{n-1}$.

(b) If $\mathcal{F}$ is a conjunction of s *non-strict* inequalities, then $\mathrm{b}(S) \leqslant (sd+1)^n$.

(c) If $\mathcal{F}$ is a conjunction of s equations and *strict* inequalities, then $\mathrm{b}(S) \leqslant \mathrm{O}(sd)^n$.

The following statement applies to more general semialgebraic sets.

Theorem 6.2 [**2**, Theorem 1; **4**, Theorem 7.38]. *If $\mathcal{F}$ is a monotone Boolean combination (that is, exclusively the connectives $\wedge$ and $\vee$ are used, with no negations) of only strict or only non-strict inequalities, then $\mathrm{b}(S) \leqslant \mathrm{O}(sd)^n$.*

In [**9**, Theorem 1], the authors proved the bound $\mathrm{b}(S) \leqslant \mathrm{O}(s^2 d)^n$ for an arbitrary Boolean formula $\mathcal{F}$. Theorem 1.10 implies the following refinement of this bound.

Theorem 6.3. *Let $\nu := \min\{k+1, n-k, s\}$. Then the kth Betti number satisfies*

$$\mathrm{b}_k(S) \leqslant \mathrm{O}(\nu s d)^n.$$

Proof. Assume first that $k > 0$. For $m = k$, construct $T(S)$ in the compactification of $\mathbb{R}^n$, as described in Section 1. The set $T(S)$ is a compact set defined by a Boolean formula with $4(k+1)s$ polynomials in $\mathbb{R}[x_1, \dots, x_n]$ of the kind $h + \delta_i$, $h - \delta_i$, $h + \varepsilon_i$, or $h - \varepsilon_i$, where $0 \leqslant i \leqslant k$, having degrees at most d. According to Lemma 1.9, there is a bijection C from the set $\mathbf{T}$ of all connected components of $T(S)$ to the set $\mathbf{S}$ of all connected components of S such that $C^{-1}(S') = T(S')$ for every $S' \in \mathbf{S}$. By Theorem 1.10(i), $\mathrm{b}_k(S') \leqslant \mathrm{b}_k(T(S'))$. It follows that

$$\mathrm{b}_k(S) = \sum_{S' \in \mathbf{S}} \mathrm{b}_k(S') \leqslant \sum_{S' \in \mathbf{S}} \mathrm{b}_k(T(S')) = \mathrm{b}_k(T(S)).$$

Then, applying the bound from Theorem 6.2 to $T(S)$, we have

$$\mathrm{b}_k(S) \leqslant \mathrm{b}_k(T(S)) \leqslant \mathrm{O}((k+1)sd)^n. \tag{6.1}$$

On the other hand, since $T(S)$ is compact, $\mathrm{b}_k(T(S)) = \mathrm{b}_{n-k-1}(\mathbb{R}^n \setminus T(S))$ by Alexander's duality. The semialgebraic set $\mathbb{R}^n \setminus T(S)$ is defined by a monotone Boolean combination of only strict inequalities, and hence, by Theorem 6.2, we have

$$\mathrm{b}_k(S) \leqslant \mathrm{b}_{n-k-1}(\mathbb{R}^n \setminus T(S)) \leqslant \mathrm{O}((n-k)sd)^n. \tag{6.2}$$

The theorem now follows from (6.1) and (6.2) and the bound $\mathrm{b}(S) \leqslant \mathrm{O}(s^2 d)^n$ from [**9**].

In the case $k = 0$, we have $\mathrm{b}_0(S) \leqslant \mathrm{b}_0(T(S))$ since the map C is surjective. Hence, by Theorem 6.2, we obtain

$$\mathrm{b}_0(S) \leqslant \mathrm{b}_0(T(S)) \leqslant \mathrm{O}(sd)^n.$$

□

6.2. *Projections of semialgebraic sets*

Let $\rho : \mathbb{R}^{n+r} \to \mathbb{R}^n$ be the projection map and $S = \{(\mathbf{x}, \mathbf{y}) \mid \mathcal{F}(\mathbf{x}, \mathbf{y})\} \subset \mathbb{R}^{n+r}$ be a semialgebraic set, where $\mathcal{F}$ is a Boolean combination of polynomial equations and inequalities of the kind $h(\mathbf{x}, \mathbf{y}) = 0$ or $h(\mathbf{x}, \mathbf{y}) > 0$, where $h \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_r]$. Suppose that the number of different polynomials h is s and that their degrees do not exceed d.

An effective quantifier elimination algorithm [**4**, Chapter 14] produces a Boolean combination $\mathcal{F}_\rho$ of polynomial equations and inequalities, with polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, defining the projection $\rho(S)$. The number of different polynomials in $\mathcal{F}_\rho$ is $(sd)^{\mathrm{O}(nr)}$ and their degrees are bounded by $d^{\mathrm{O}(r)}$. Then Theorem 6.3 (or [**9**, Theorem 1]) implies that

$$\mathrm{b}_k(\rho(S)) \leqslant (sd)^{\mathrm{O}(n^2 r)} \tag{6.3}$$

for any $k \geqslant 0$. We now improve this bound as follows.

Theorem 6.4. *The kth Betti number of $\rho(S)$ satisfies the inequality*

$$\mathrm{b}_k(\rho(S)) \leqslant \sum_{0 \leqslant p \leqslant k} \mathrm{O}((p+1)(k+1)sd)^{n+(p+1)r} \leqslant ((k+1)sd)^{\mathrm{O}(n+kr)}.$$

Proof. For $k = 0$, the bound immediately follows from Theorem 6.3. So assume that $k > 0$. The set S is represented by the families $\{S_\delta\}_\delta$ and $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon}$ in the compactification of $\mathbb{R}^{n+r}$ as described in Section 1. According to Lemma 1.3, the projection $\rho(S)$ is represented by the families $\{\rho(S_\delta)\}_\delta$ and $\{\rho(S_{\delta,\varepsilon})\}_{\delta,\varepsilon}$ in the compactification of $\mathbb{R}^n$. Fix $m = k$. Then the set $T(\rho(S)) = \rho(T(S))$ is defined. According to Corollary 2.15, we have

$$\mathrm{b}_k(\rho(T(S))) \leqslant \sum_{p+q=k} \mathrm{b}_q(W_p),$$

where

$$W_p = \underbrace{T(S) \times_{\rho(T(S))} \cdots \times_{\rho(T(S))} T(S)}_{p+1 \text{times}}.$$

The fibre product $W_p \subset \mathbb{R}^{n+(p+1)r}$ is definable by a Boolean formula with

$$4(p+1)(k+1)s$$

polynomials of degrees not exceeding d. Hence, by Theorem 6.2, we have

$$\mathrm{b}_q(W_p) \leqslant \mathrm{O}((p+1)(k+1)sd)^{n+(p+1)r}.$$

It follows that

$$\mathrm{b}_k(T(\rho(S))) \leqslant \sum_{0 \leqslant p \leqslant k} \mathrm{O}((p+1)(k+1)sd)^{n+(p+1)r} \leqslant ((k+1)sd)^{\mathrm{O}(n+kr)}. \tag{6.4}$$

Finally, by Theorem 1.10(i), $\mathrm{b}_k(\rho(S)) \leqslant \mathrm{b}_k(T(\rho(S)))$, which, in conjunction with (6.4), completes the proof. □

6.3. *Semi- and sub-Pfaffian sets*

Necessary definitions regarding semi-Pfaffian and sub-Pfaffian sets can be found in [**7**, **8**] (see also [**11**]).

According to [**16**, **18**], $\mathbb{R}$ admits an o-minimal expansion $\mathcal{P}(\mathbb{R})$ in which sub-Pfaffian sets are definable. In what follows we adopt $\mathcal{P}(\mathbb{R})$ as the o-minimal structure containing all S, S_δ, and $S_{\delta,\varepsilon}$.

Let $S = \{\mathbf{x} \mid \mathcal{F}(\mathbf{x})\} \subset (0,1)^n$ be a semi-Pfaffian set, where $\mathcal{F}$ is a Boolean combination of equations and inequalities with s different Pfaffian functions (here and in what follows $(0,1)$ can be replaced by any, bounded or unbounded, interval). Assume that all functions are defined in $(0,1)^n$, have a common Pfaffian chain of order ℓ, and have degree (α, β). A straightforward generalization of Theorem 6.2 gives the following upper bound.

THEOREM 6.5 [**8**, Theorem 3.4; **19**, Theorem 1]. *If $\mathcal{F}$ is a monotone Boolean combination of only strict or only non-strict inequalities such that $\overline{S} \subset (0,1)^n$, then*

$$\mathrm{b}(S) \leqslant s^n 2^{\ell(\ell-1)/2} \mathrm{O}(n\beta + \min\{n,\ell\}\alpha)^{n+\ell}.$$

In conjunction with Theorem 1.10, this implies the following bound for the set S defined by an *arbitrary* Boolean formula $\mathcal{F}$.

THEOREM 6.6. *Let $\nu := \min\{k+1, n-k, s\}$. Then the kth Betti number*

$$\mathrm{b}_k(S) \leqslant (\nu s)^n 2^{\ell(\ell-1)/2} \mathrm{O}(n\beta + \min\{n,\ell\}\alpha)^{n+\ell}.$$

The proof of the theorem is analogous to the proof of Theorem 6.3.

REMARK 6.7. Unlike Theorem 6.5, the condition $\overline{S} \subset (0,1)^n$ is not required in Theorem 6.6, since taking the conjunction of inequalities $0 < x_i < 1$, for $i = 1, \dots, n$, with $\mathcal{F}$ guarantees that the closed set $T(S) \subset (0,1)^n$.

Now we consider the sub-Pfaffian case. Let $\rho : \mathbb{R}^{n+r} \to \mathbb{R}^n$ be the projection map and $S = \{(\mathbf{x}, \mathbf{y}) \mid \mathcal{F}(\mathbf{x}, \mathbf{y})\} \subset (0,1)^{n+r}$ be a semi-Pfaffian set, where $\mathcal{F}$ is a Boolean combination of Pfaffian equations and inequalities. Suppose that all different Pfaffian functions occurring in $\mathcal{F}$ are defined in $(0,1)^{n+r}$, have a common Pfaffian chain of order ℓ, their number is s, and their degree is (α, β). Since the Pfaffian o-minimal structure does not admit quantifier elimination (that is, the projection of a semi-Pfaffian set may not be semi-Pfaffian; see [**8**]), it is not possible to apply in the Pfaffian case the same method that we used to obtain the bound (6.4). On the other hand, the method employed in the proof of Theorem 6.4 extends straightforwardly to projections of semi-Pfaffian sets and produces the following first general singly exponential upper bound for Betti numbers of sub-Pfaffian sets.

THEOREM 6.8. *The kth Betti number of $\rho(S)$ satisfies the inequality*

$$\mathrm{b}_k(\rho(S)) \leqslant (ks)^{\mathrm{O}(n+(k+1)r)} 2^{\mathrm{O}(k\ell)^2} ((n+(k+1)r)(\alpha+\beta))^{n+(k+1)r+k\ell}.$$

The proof of the theorem is analogous to the proof of Theorem 6.4.

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References

1. A. Barvinok, 'On the Betti numbers of semi-algebraic sets defined by few quadratic inequalities', *Math. Z.* 225 (1997) 231–244.
2. S. Basu, 'On bounding the Betti numbers and computing the Euler characteristics of semialgebraic sets', *Discrete Comput. Geom.* 22 (1999) 1–18.
3. S. Basu, D. Pasechnik and M.-F. Roy, 'Bounding the Betti numbers and computing the Euler–Poincaré characteristic of semi-algebraic sets defined by partly quadratic systems of polynomials', *J. Eur. Math. Soc.*, to appear. Preprint, 2007, arXiv:0708.3522.
4. S. Basu, R. Pollack and M.-F. Roy, *Algorithms in real algebraic geometry*, 2nd edn (Springer, Berlin, 2006).
5. A. Björner, 'Nerves, fibers and homotopy groups', *J. Combin. Theory* Ser. A 102 (2003) 88–93.
6. M. Coste, 'An introduction to O-minimal geometry', Dottorato di Ricerca in Matematica, Dip. Mat. Univ. Pisa, Istituti Editoriale e Poligrafici Internazionali, Pisa, 2000.
7. A. Gabrielov and N. Vorobjov, 'Complexity of cylindrical decompositions of sub-Pfaffian sets', *J. Pure Appl. Algebra* 164 (2001) 179–197.
8. A. Gabrielov and N. Vorobjov, 'Complexity of computations with Pfaffian and Noetherian functions', *Normal forms, bifurcations and finiteness problems in differential equations* (eds Yu. Ilyashenko and C. Rousseau; Kluwer, Dordrecht, 2004) 211–250.
9. A. Gabrielov and N. Vorobjov, 'Betti numbers of semialgebraic sets defined by quantifier-free formulae', *Discrete Comput. Geom.* 33 (2005) 395–401.
10. A. Gabrielov, N. Vorobjov and T. Zell, 'Betti numbers of semialgebraic and sub-Pfaffian sets', *J. London Math. Soc.* 69 (2004) 27–43.
11. A. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs 88 (American Mathematical Society, Providence, RI, 1991).
12. J. Milnor, 'On the Betti numbers of real varieties', *Proc. Amer. Math. Soc.* 15 (1964) 275–280.
13. O. A. Oleinik, 'Estimates of the Betti numbers of real algebraic hypersurfaces', *Mat. Sb.* 28 (1951) 635–640 (Russian).
14. I. G. Petrovskii and O. A. Oleinik, 'On the topology of real algebraic hypersurfaces', *Izv. Acad. Nauk SSSR* 13 (1949) 389–402 (Russian), *Amer. Math. Soc. Transl.* 7 (1962) 399–417 (English).
15. E. Spanier, *Algebraic topology* (Springer, New York, 1981).
16. P. Speissegger, 'The Pfaffian closure of an o-minimal structure', *J. Reine Angew. Math.* 508 (1999) 189–211.
17. R. Thom, 'Sur l'homologie des variétés algebriques réelles', *Differential and combinatorial topology* (ed. S. S. Cairns; Princeton University Press, Princeton, NJ, 1965) 255–265.
18. A. J. Wilkie, 'A theorem of the complement and some new o-minimal structures', *Selecta. Math.* (*N.S.*) 5 (1999) 397–421.
19. T. Zell, 'Betti numbers of semi-Pfaffan sets', *J. Pure Appl. Algebra* 139 (1999) 323–338.

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