

# Fundamentals of continuum percolation

Mathew D. Penrose

*University of Bath, UK*

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## CONTINUUM PERCOLATION (see Meester and Roy 1996)

Let  $\mathcal{P}_\lambda$  be a homogeneous Poisson point process in  $\mathbf{R}^d$  with intensity  $\lambda$ , i.e.

$$\mathcal{P}_\lambda(A) \sim \text{Poisson}(|A|)$$

and  $\mathcal{P}_\lambda(A_i)$  are independent variables for  $A_1, A_2, \dots$  disjoint.

**Gilbert Graph.** Form a graph  $\mathcal{G}_\lambda := G(\mathcal{P}_\lambda)$  on  $\mathcal{P}_\lambda$  by connecting two Poisson points  $x, y$  iff  $|x - y| \leq 1$ .

Form graph  $\mathcal{G}_\lambda^0 := G(\mathcal{P}_\lambda \cup \{0\})$  similarly on  $\mathcal{P}_\lambda \cup \{0\}$ .

Let  $p_k(\lambda)$  be the prob. that the component of  $\mathcal{G}_\lambda^0$  containing  $\mathbf{0}$  has  $k$  vertices (also depends on  $d$ ).

$p_\infty(\lambda) := 1 - \sum_{k=0}^{\infty} p_k(\lambda)$  be the prob. that this component is infinite.

## RANDOM GEOMETRIC GRAPHS (finite analog) (Penrose 2003)

Let  $U_1, \dots, U_n$  be independently uniformly randomly scattered in a cube of volume  $n/\lambda$  in  $d$ -space. Form a graph  $\mathcal{G}_{n,\lambda}$  on  $\{1, 2, \dots, n\}$  by

$$i \sim j \quad \text{iff} \quad |U_i - U_j| \leq 1$$

$\mathcal{G}_{n,\lambda}$  is a geometric alternative to the classical random graph model. Define  $|C_i|$  to be the order of the component of  $\mathcal{G}_{n,\lambda}$  containing  $i$ . It can be shown that  $n^{-1} \sum_{i=1}^n \mathbf{1}\{|C_i| = k\} \xrightarrow{P} p_k(\lambda)$  as  $n \rightarrow \infty$ , i.e. for large  $n$ ,

$$P \left[ n^{-1} \sum_{i=1}^n \mathbf{1}\{|C_i| = k\} \approx p_k(\lambda) \right] \approx 1$$

That is, the proportionate number of vertices of  $G_n$  lying in components of order  $k$ , converges to  $p_k(\lambda)$  in probability.

A FORMULA FOR  $p_k(\lambda)$ .

$$p_{k+1}(\lambda) = (k + 1)\lambda^k \int_{(\mathbf{R}^d)^k} h(x_1, \dots, x_k) \\ \times \exp(-\lambda A(\mathbf{0}, x_1, \dots, x_k)) dx_1 \dots dx_k$$

where  $h(x_1, \dots, x_k)$  is 1 if  $G(\{\mathbf{0}, x_1, \dots, x_k\})$  is connected and  $0 \prec x_1 \prec \dots \prec x_k$  lexicographically, otherwise zero;

and  $A(\mathbf{0}, x_1, \dots, x_k)$  is the volume of the union of 1-balls centred at  $0, x_1, \dots, x_k$ .

Not tractable for large  $k$ .

## THE PHASE TRANSITION

If  $p_\infty(\lambda) = 0$ , then  $\mathcal{G}_\lambda$  has no infinite component, almost surely.

If  $p_\infty(\lambda) > 0$ , then  $\mathcal{G}_\lambda$  has a unique infinite component, almost surely.

Also,  $p_\infty(\lambda)$  is nondecreasing in  $\lambda$ .

Fundamental theorem: If  $d \geq 2$  then

$$\lambda_c(d) := \sup\{\lambda : p_\infty(\lambda) = 0\} \in (0, \infty).$$

If  $d = 1$  then  $\lambda_c(d) = \infty$ . From now on, assume  $d \geq 2$ . The value of  $\lambda_c(d)$  is not known.

## Sketch proof of fundamental theorem.

Be wise, discretize! Divide  $\mathbf{R}^d$  into boxes of side  $\varepsilon$  ( $\varepsilon$  small, fixed).

If  $|C_0| = \infty$  there is an infinite path through occupied boxes successively distance less than two from each other.

There exists finite  $\gamma$  such that the expected number of such paths of length  $n$  is at most  $\gamma^n(1 - e^{-\lambda\varepsilon^d})^n$ . For  $\lambda$  small enough, this tends to zero as  $n \rightarrow \infty$ . Hence the probability of an infinite path is zero.

Conversely, if  $|C_0|$  is finite, there must be a path of empty boxes surrounding the origin and by a similar argument, can show for  $\lambda$  large enough the probability of this happening is less than 1.

## CONTINUITY OF $p_\infty(\lambda)$

Clearly  $p_\infty(\lambda) = 0$  for  $\lambda < \lambda_c$

Also  $p_\infty(\lambda)$  is increasing in  $\lambda$  on  $\lambda > \lambda_c$ .

Less trivially, it is known that  $p_\infty(\lambda)$  is continuous in  $\lambda$  on  $\lambda \in (\lambda_c, \infty)$  and is *right* continuous at  $\lambda = \lambda_c$ , i.e.

$$p_\infty(\lambda_c) = \lim_{\lambda \downarrow \lambda_c} p_\infty(\lambda).$$

So  $p_\infty(\cdot)$  is continuous on  $(0, \infty)$  iff

$$p_\infty(\lambda_c) = 0.$$

This is known to hold for  $d = 2$  (Alexander 1996) and for large  $d$  (Tanemura 1996). It is conjectured to hold for all  $d$ .

## HIGH-INTENSITY ASYMPTOTICS

Let  $\theta$  denote the volume of the unit ball in  $\mathbf{R}^d$ . As  $\lambda \rightarrow \infty$ ,  $p_\infty(\lambda) \rightarrow 1$  and in fact (Penrose 1991)

$$1 - p_\infty(\lambda) \sim p_1(\lambda) = \exp(-\lambda\theta)$$

This says that for large  $\lambda$ , given the unlikely event that the component of  $\mathcal{G}_\lambda^0$  containing  $\mathbf{0}$  is finite, it is likely to consist of an isolated vertex. Also, for  $k \geq 1$ ,

$$-\log p_{k+1}(\lambda) = \lambda\theta + (d-1)k \log \frac{\lambda}{k} + O(1)$$

as  $\lambda \rightarrow \infty$  with  $k$  fixed (Alexander 1991).



## LARGE- $k$ ASYMPTOTICS FOR $p_k(\lambda)$

Suppose  $\lambda < \lambda_c$ . Then there exists  $\zeta(\lambda) > 0$  such that

$$\zeta(\lambda) = \lim_{n \rightarrow \infty} (-n^{-1} \log p_n(\lambda))$$

or more informally,  $p_n(\lambda) \sim (e^{-\zeta(\lambda)})^n$ .

Proof uses subadditivity. With  $\mathbf{x}_1^n := (x_1, \dots, x_n)$ , recall

$$p_{n+1}(\lambda) = (n+1)\lambda^n \int h(\mathbf{x}_1^n) e^{-\lambda A(\mathbf{0}, \mathbf{x}_1^n)} d\mathbf{x}_1^n.$$

Setting  $q_n := p_{n+1}/(n+1)$ , can show  $q_n q_m \leq q_{n+m-1}$ , so  $-\log q_n/(n-1) \rightarrow \inf_{n \geq 1} (-\log q_n/(n-1)) := \zeta$  as  $n \rightarrow \infty$ .

That  $\zeta(\lambda) > 0$  is a deeper result.

## LARGE- $k$ ASYMPTOTICS: THE SUPERCRITICAL CASE

Suppose  $\lambda > \lambda_c$ . Then

$$\limsup_{n \rightarrow \infty} \left( n^{-(d-1)/d} \log p_n(\lambda) \right) < 0$$

$$\liminf_{n \rightarrow \infty} \left( n^{-(d-1)/d} \log p_n(\lambda) \right) > -\infty$$

Loosely speaking, this says that in the supercritical case  $p_n(\lambda)$  decays exponentially in  $n^{1-1/d}$ , whereas in the subcritical case it decays exponentially in  $n$ .

## HIGH DIMENSIONAL ASYMPTOTICS (Penrose 1996)

Let  $\theta_d$  denote the volume of the unit ball in  $\mathbf{R}^d$ . Suppose  $\lambda = \lambda(d)$  is chosen so that  $\lambda(d)\theta_d = \mu$  (this is the expected degree of the origin  $\mathbf{0}$  in  $\mathcal{G}_\lambda^0$ ).

Asymptotically as  $d \rightarrow \infty$  with  $\mu$  fixed, the structure of  $C_0$  converges to that of a branching process  $(Z_0, Z_1, \dots)$  with Poisson ( $\mu$ ) offspring distribution ( $Z_0 = 1$ ,  $Z_n$  is  $n$ th generation size). So (Penrose 1996):

$$p_k(\lambda(d)) \rightarrow \tilde{p}_k(\mu) := P[\sum_{n=0}^{\infty} Z_n = k];$$

$$p_\infty(\lambda(d)) \rightarrow \psi(\mu) := P[\sum_{n=0}^{\infty} Z_n = \infty];$$

$$\theta_d \lambda_c(d) \rightarrow 1.$$

## DETAILS OF THE BRANCHING PROCESS THEORY:

$$\tilde{p}_k(\mu) = P\left[\sum_{n=0}^{\infty} Z_n = k\right] = \frac{k^{k-2}}{(k-1)!} \mu^{k-1} e^{-k\mu}.$$

$t = 1 - \psi(\mu)$  is the smallest positive solution to  $t = \exp(\mu(t - 1))$ . In particular,

$$\psi(\mu) = 0, \quad \text{if } \mu \leq 1$$

$$\psi(\mu) > 0, \quad \text{if } \mu > 1$$

## LARGE COMPONENTS FOR THE RGG

Consider again the random geometric graph  $\mathcal{G}_{n,\lambda}$  (on  $n$  uniform random points in a cube of volume  $n/\lambda$  in  $d$ -space)

Let  $L_1(\mathcal{G}_{n,\lambda})$  be the size of the largest component, and  $L_2(\mathcal{G}_{n,\lambda})$  the size of the second largest component ('size' measured by number of vertices). As  $n \rightarrow \infty$  with  $\lambda$  fixed:

if  $\lambda > \lambda_c$  then  $n^{-1}L_1(\mathcal{G}_{n,\lambda}) \xrightarrow{P} p_\infty(\lambda) > 0$

if  $\lambda < \lambda_c$  then  $(\log n)^{-1}L_1(\mathcal{G}_{n,\lambda}) \xrightarrow{P} 1/\zeta(\lambda)$

and for the Poissonized RGG  $\mathcal{G}_{N_n,\lambda}$  ( $N_n \sim \text{Poisson}(n)$ ),

$L_2(\mathcal{G}_{N_n,\lambda}) = O(\log n)^{d/(d-1)}$  in probability if  $\lambda > \lambda_c$

**CENTRAL LIMIT THEOREMS.** Let  $K(\mathcal{G}_{n,\lambda})$  be the number of components of  $\mathcal{G}_{n,\lambda}$ . As  $n \rightarrow \infty$  with  $\lambda$  fixed,

$$P \left[ \frac{K(\mathcal{G}_{n,\lambda}) - \mathbf{E}K(\mathcal{G}_{n,\lambda})}{\sigma\sqrt{n}} \leq t \right] \rightarrow \Phi(t) := (2\pi)^{-1/2} \int_{-\infty}^t e^{-x^2/2} dx$$

and if  $\lambda > \lambda_c$ ,

$$P \left[ \frac{L_1(\mathcal{G}_{n,\lambda}) - \mathbf{E}L_1(\mathcal{G}_{n,\lambda})}{\tau\sqrt{n}} \leq t \right] \rightarrow \Phi(t).$$

Here  $\sigma$  and  $\tau$  are positive constants, dependent on  $\lambda$ .

**ISOLATED VERTICES.** Suppose  $d = 2$ . Let  $N_0(\mathcal{G}_{n,\lambda})$  be the number of isolated vertices. The expected number of isolated vertices satisfies

$$\mathbf{E}N_0(\mathcal{G}_{n,\lambda}) \approx n \exp(-\pi\lambda)$$

so if we fix  $t$  and take  $\lambda(n) = (\log n + t)/\pi$ , then as  $n \rightarrow \infty$ ,

$$\mathbf{E}N_0(\mathcal{G}_{n,\lambda(n)}) \rightarrow e^{-t}.$$

Also,  $N_0$  is approximately Poisson distributed so

$$P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0] \rightarrow \exp(-e^{-t}).$$

**CONNECTIVITY.** Note  $\mathcal{G}_{n,\lambda}$  is connected iff  $K(\mathcal{G}_{n,\lambda}) = 1$ .

Clearly  $P[K(\mathcal{G}_{n,\lambda}) = 1] \leq P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0]$ . Again taking  $\lambda(n) = (\log n + t)/\pi$  with  $t$  fixed, it turns out (Penrose 1997) that

$$\lim_{n \rightarrow \infty} P[K(\mathcal{G}_{n,\lambda(n)}) = 1] = \lim_{n \rightarrow \infty} P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0] = \exp(-e^{-t})$$

or in other words,

$$\lim_{n \rightarrow \infty} (P[K(\mathcal{G}_{n,\lambda(n)}) > 1] - P[N_0(\mathcal{G}_{n,\lambda(n)}) > 0]) = 0.$$

This is related to the earlier result that at high intensity, the most likely way for a vertex to be in a finite component is if it is isolated.



## THE CONNECTIVITY THRESHOLD

Let  $V_1, \dots, V_n$  be independently uniformly randomly scattered in  $[0, 1]^d$ . Form a graph  $\mathcal{G}_n^r$  on  $\{1, 2, \dots, n\}$  by

$$i \sim j \quad \text{iff} \quad |V_i - V_j| \leq r.$$

Given the values of  $V_1, \dots, V_n$ , define the connectivity threshold  $\rho_n(K = 1)$ , and the no-isolated-vertex threshold  $\rho_n(N_0 = 0)$ , by

$$\begin{aligned} \rho_n(K = 1) &= \min\{r : K(\mathcal{G}_n^r) = 1\}; \\ \rho_n(N_0 = 0) &= \min\{r : N_0(\mathcal{G}_n^r) = 1\}. \end{aligned}$$

The preceding result can be interpreted as giving the limiting distributions of these thresholds (suitably scaled and centred) as  $n \rightarrow \infty$ : they have the same limiting behaviour.

Taking  $\lambda(n) = (\log n + t)/\pi$  with  $t$  fixed, we have

$$\begin{aligned} P[K(\mathcal{G}_{n,\lambda(n)}) = 1] &= P[K(\mathcal{G}_n^{\sqrt{\lambda(n)/n}}) = 1] \\ &= P\left[\rho_n(K = 1) \leq \sqrt{(\log n + t)/(\pi n)}\right] \end{aligned}$$

so the earlier result

$$\lim_{n \rightarrow \infty} P[K(\mathcal{G}_{n,\lambda(n)}) = 1] = \lim_{n \rightarrow \infty} P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0] = \exp(-e^{-t})$$

implies

$$\lim_{n \rightarrow \infty} P[n\pi(\rho_n(K = 1))^2 - \log n \leq t] = \exp(e^{-t})$$

and likewise for  $\rho_n(N_0) = 0$ . In fact we have a stronger result:

$$\lim_{n \rightarrow \infty} P[\rho_n(K = 1) = \rho_n(N_0 = 0)] = 1.$$

**MULTIPLE CONNECTIVITY.** Given  $k \in \mathbf{N}$ , a graph is  $k$ -connected if for any two distinct vertices there are  $k$  disjoint paths connecting them.

Let  $\rho_{n,k}$  be the smallest  $r$  such that  $G_n^r$  is  $k$ -connected.

Let  $\rho_n(N_{<k} = 0)$  be the smallest  $r$  such that  $\mathcal{G}_n^r$  has no vertex of degree less than  $k$  (a random variable determined by  $V_1, \dots, V_n$ ).

Then

$$\lim_{n \rightarrow \infty} P[\rho_{n,k} = \rho_n(N_{<k} = 0)] = 1.$$

The limit distribution of  $\rho_n(N_{<k} = 0)$  can be determined via Poisson approximation, as with  $\rho_n(N_0 = 0)$ .

## HAMILTONIAN PATHS

Let  $\rho_n(\text{Ham})$  be the smallest  $r$  such that  $\mathcal{G}_n^r$  has a Hamiltonian path (i.e. a self-avoiding tour through all the vertices). Clearly

$$\rho_n(\text{Ham}) \geq \rho_n(N_{<2} = 0).$$

It has recently been established (Balogh, Bollobas, Walters, Krivelevich, Müller 2009) that

$$\lim_{n \rightarrow \infty} P[\rho_n(\text{Ham}) = \rho_n(N_{<2} = 0)] = 1.$$

## OTHER PROPERTIES OF RGGs

Not so much connection to percolation but asymptotic behaviour of other quantities arising from of random geometric graph have been considered, including

The largest and smallest degree.

The number of edges, number of triangles and so on.

The clique and chromatic number.

In some cases, non-uniform distributions of the vertices  $V_1, \dots, V_n$  have been considered.

## OTHER MODELS OF CONTINUUM PERCOLATION

- **The Boolean model.** Let  $\Xi$  be the union of balls of random radius centred at the points of  $\mathcal{P}_\lambda$  (with some specified radius distribution). Look at the connected components of  $\Xi$ .
- **The random connection model.** Given  $\mathcal{P}_\lambda$ , let each pair  $(x, y)$  of points of  $\mathcal{P}_\lambda$  be connected with probability  $f(|x - y|)$  with  $f$  some specified function satisfying

$$\int_{\mathbf{R}^d} f(|x|) dx < \infty$$

Both of these are generalizations of the model we have been considering.

- **Nearest-neighbour percolation.** Fix  $k \in \mathbf{N}$ . Join each point of  $\mathcal{P}_\lambda$  by an undirected edge to its  $k$  nearest neighbours in  $\mathcal{P}_\lambda$ . Look at components of resulting graph.
- **Lilypond model.** At time zero, start growing a ball of unit rate outwards from each point of  $\mathcal{P}_\lambda$ , stopping as soon as it hits another ball. This yields a maximal system of non-overlapping balls on  $\mathcal{P}$ .
- **Voronoi cell percolation.** Let each vertex of  $\mathcal{P}_\lambda$  be coloured red (with probability  $p$ ), otherwise blue. Look at connectivity properties of the union of red Voronoi cells.

## References

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