Connectivity of soft random geometric graphs

Mathew Penrose
(University of Bath)

International Conference
Geometry and Physics of Spatial Random Systems
Freudenstadt
September 2013

1. A graph $G = (V, E)$ is said to be **connected** if for all $v, w \in V$ $(v \neq w)$ there is a path of edges from v to w.

2. An $\mathbb{N}$-valued random variable X has **expected value** $\mathbb{E}[X]$ defined by

$$\mathbb{E}[X] = \sum_{n \geq 1} n \mathbb{P}[X = n]$$

where $\mathbb{P}$ denotes probability.

3. If X is a *Poisson* distributed random variable with parameter $\mu > 0$ then $\mathbb{E}[X] = \mu$ and $\mathbb{P}[X = 0] = e^{-\mu}$.

4. If X, Y are random variables, then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Overview

Let $\mathcal{K}$ be the set of all connected finite graphs with at least two vertices. Let $\mathcal{M}_1$ be the set of all graphs G with minimum degree at least 1, i.e. with $N_0(G) = 0$ where $N_0(G)$ is the number of isolated vertices. Clearly

$$\mathcal{K} \subset \mathcal{M}_1.$$

Our main message is that for certain types of large *random* graphs G,

$$\mathbb{P}[G \in \mathcal{K}] \approx \mathbb{P}[G \in \mathcal{M}_1]$$

and also $N_0(G)$ is approximately Poisson so

$$\mathbb{P}[G \in \mathcal{M}_1] \approx \exp(-\mathbb{E}[N_0(G)])$$

and therefore

$$\mathbb{P}[G \in \mathcal{K}] \approx \exp(-\mathbb{E}[N_0(G)]).$$

The Erdos-Renyi random graph $G(n,p)$

Let $n \in \mathbb{N}$ and $p \in (0,1)$. The random graph $G(n,p)$ has n vertices; each of the $n(n-1)/2$ possible edges is included with probability p (independently). It is not hard to see that

$$\mathbb{E}[N_0(G(n,p))] = n(1-p)^{n-1} \approx ne^{-np}$$

and if we choose $p = p_n$ so that $ne^{-np_n} \to \beta$ as $n \to \infty$, it turns out that $N_0(G(n,p_n)$ is approximately *Poisson* for n large, so

$$\mathbb{P}[G(n,p_n) \in \mathcal{M}_1] = \mathbb{P}[N_0(G(n,p_n)) = 0] \to e^{-\beta}.$$

Then it is easy to deduce that

$$\lim_{n\to\infty} \left(\sup_{0 \le p \le 1} |\mathbb{P}[G(n,p) \in \mathcal{M}_1] - \exp(-ne^{-np})| \right) = 0$$

Equivalence of '$G \in \mathcal{K}$' and '$G \in \mathcal{M}_1$' for $G(n,p)$

Erdos and Renyi (1959) proved that if we choose p_n so $n \exp(-np_n) \to \beta$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathbb{P}[G(n,p_n) \in \mathcal{K}] = e^{-\beta}.$$

Then we can deduce that

$$\lim_{n\to\infty} \left(\sup_{0\le p\le 1} |\mathbb{P}[G(n,p) \in \mathcal{K}] - \exp(-ne^{-np})| \right) = 0$$

and

$$\lim_{n\to\infty} \sup_{0\le p\le 1} \mathbb{P}[G(n,p_n) \in \mathcal{M}_1 \setminus \mathcal{K}] = 0.$$

The random geometric (Gilbert) graph $G(n,r)$

Let $r > 0, n \in \mathbb{N}$. The graph $G(n,r)$ has vertex set $V_n = \{X_1, \ldots, X_n\}$ with X_i independently uniformly distributed over the unit square $[0,1]^2$. The edges consist of those $\{X_i, X_j\}$ such that $|X_i - X_j| \leq r$, where $|\cdot|$ is the Euclidean norm.

Sometimes more convenient to consider $G(N_n, r)$ where N_n is Poisson (n), but results are similar - we'll ignore this distinction.

Modulo boundary effects, for r small we have

$$\mathbb{E}[N_0(G(n,r))] = n(1-\pi r^2)^{n-1} \approx n \exp(-\pi n r^2).$$

If we choose r_n so $n \exp(-\pi n r_n^2) \to \beta \in \mathbb{R}$ then we have another Poisson approximation result (Dette and Henze 1989):

$$\lim_{n\to\infty} \mathbb{P}[N_0(G(n,r_n)) = 0] = \exp(-\beta).$$

Hence,

$$\lim_{n\to\infty} \sup_{0 \leq r \leq \sqrt{2}} \left(|\mathbb{P}[N_0(G(n,r)) = 0] - \exp(-n \exp(-\pi n r^2))| \right) = 0.$$

Equivalence result for $G(n, r)$

If we choose r_n so $n \exp(-n\pi r_n^2)] \to \beta$, then

$$\lim_{n\to\infty} \mathbb{P}[G(n, r_n) \in \mathcal{K}] = e^{-\beta}.$$

We can then deduce that

$$\lim_{n\to\infty} \left(\sup_{r\geq 0} \left| \mathbb{P}[G(n, r) \in \mathcal{K}] - \exp(-n \exp(-n\pi r^2)) \right| \right) = 0$$

and

$$\lim_{n\to\infty} \sup_{r\geq 0} \mathbb{P}[G(n, r) \in \mathcal{M}_1 \setminus \mathcal{K}] = 0.$$

The proof of these results is entirely different from in the case of $G(n, p)$. Essentially they are due to MP (1997); see also Gupta and Kumar (1998), who were interested in wireless communications networks.

Multiple connectivity, and thresholds

Let $k \in \mathbb{N}$. We say a graph is k-connected if for any two vertices x, y there exist k disjoint paths from x to y. Let $\mathcal{K}_k$ be the set of k-connected graphs with at least two vertices. Clearly $\mathcal{K}_k \subset \mathcal{M}_k$, the set of graphs of minimum degree at least k. MP (1999) showed the equivalence

$$\lim_{n\to\infty} \sup_{r>0} \mathbb{P}[G(n,r) \in \mathcal{M}_k \setminus \mathcal{K}_k] = 0.$$

Given a realization of $V_n = \{X_1, \ldots, X_n\}$, the *threshold radius* for k-connectivity, respectively the threshold radius for minimum degree at least k, are defined by

$$\rho_n(\mathcal{K}_k) = \inf\{r : G(n,r) \in \mathcal{K}_k\};$$
$$\rho_n(\mathcal{M}_k) = \inf\{r : G(n,r) \in \mathcal{M}_k\}.$$

These are random variables, determined by the configuration V_n.

Strong equivalence

Let $k \in \mathbb{N}$. Since $\mathcal{K}_k \subset \mathcal{M}_k$ we have that $\rho_n(\mathcal{K}_k) \geq \rho_n(\mathcal{M}_k)$. A strong version of the preceding equivalence holds (MP 1999), namely

$$\lim_{n\to\infty} \mathbb{P}[\rho_n(\mathcal{K}_k) = \rho_n(\mathcal{M}_k)] = 1.$$

That is, if we add the edges amongst V_n one by in order of increasing length, the graph becomes k-connected at the same time as the minimum degree goes above $k-1$. In particular

$$\lim_{n\to\infty} \mathbb{P}[\rho_n(\mathcal{K}) = \rho_n(\mathcal{M}_1)] = 1.$$

It is not known if there is a random finite N such that

$$\mathbb{P}[\rho_n(\mathcal{K}) = \rho_n(\mathcal{M}_1), \quad \forall n \geq N] = 1.$$

The Geometric Erdos-Renyi graph

Given $n \in \mathbb{N}$, $p \in [0,1]$, and $r > 0$, let $G(n,r,p)$ be the graph with vertex set V_n (n independent uniform in $[0,1]^2$) and with vertices x, y connected with probability p whenever $|x-y| \leq r$ (and not connected whenever $|x-y| > r$). This is the intersection of the graphs $G(n,p)$ and $G(n,r)$. If $(r_n, p_n)_{n \geq 1}$ chosen so that $p_n \gg 1/\log n$ as $n \to \infty$, and $n \exp(-n\pi p_n r_n^2) \to \beta$, then (Yi et al. 2006, MP 2013+)

$$\lim_{n\to\infty} \mathbb{P}[G(n,r_n,p_n) \in \mathcal{K}] = \lim_{n\to\infty} \mathbb{P}[G(n,r_n,p_n) \in \mathcal{M}_1] = \exp(-\beta).$$

If $p_n = O(1/\log n)$ the second equality above fails because of boundary effects. However, for *any* $(r_n, p_n)_{n \geq 1}$ we have (MP 2013+)

$$\lim_{n\to\infty} \mathbb{P}[G(n,r_n,p_n) \in \mathcal{M}_1 \setminus \mathcal{K}] = 0$$

or to put this another way:

$$\lim_{n\to\infty} \sup_{r>0, p\in(0,1]} \mathbb{P}[G(n,r,p) \in \mathcal{M}_1 \setminus \mathcal{K}] = 0.$$

More choices of (r_n, p_n)

If $\frac{n^{-1/3}}{\log n} \ll p_n \ll \frac{1}{\log n}$ and

$$e^{-n\pi p_n r_n^2/2}(n/(p_n \log(n/p_n)))^{1/2} \to \frac{\beta}{2\sqrt{\pi}},$$

or if $p_n \ll 1/\log n$ and $\sup\{r_n : n \in \mathbb{N}\} < 1$ then

$$e^{-n\pi p_n r_n^2/4}/(p_n \log 1/p_n) \to \beta/\pi$$

then (MP2013+)

$$\lim_{n\to\infty} \mathbb{P}[G(n, r_n, p_n) \in \mathcal{K}] = \lim_{n\to\infty} \mathbb{P}[G(n, r_n, p_n) \in \mathcal{M}_1] = \exp(-\beta).$$

Thresholds for Geometric Erdos-Renyi graphs

Given n and p, suppose we have the random point set V_n and also independent variables $I_{ij}, 1 \leq i < j \leq n$ with $\mathbb{P}[I_{ij} = 1] = p$ and $\mathbb{P}[I_{i,j} = 0] = 1 - p$. We can take the edge set of $G(n, r, p)$ to be

$$E(G(n, r, p)) = \{\{i, j\} : |X_i - X_j| \leq r, I_{ij} = 1\}.$$

Define the random variables (determined by V_n and $\{I_{ij}, 1 \leq i < j \leq n\}$):

$$\rho_{n,p}(\mathcal{K}) = \min\{r : G(n, r, p) \in \mathcal{K}\};$$
$$\rho_{n,p}(\mathcal{M}_1) = \min\{r : G(n, r, p) \in \mathcal{M}_1\}.$$

Then (MP 2013+) for any choice of $(p_n)_{n \in \mathbb{N}}$,

$$\lim_{n \to \infty} \mathbb{P}[\rho_{n,p_n}(\mathcal{K}) = \rho_{n,p_n}(\mathcal{M}_1)] = 1.$$

'Soft' random geometric graphs

Given a nonincreasing connection function $\phi : \mathbb{R}_+ \to [0,1]$, consider the graph $G(n,\phi)$ with vertex set $V_n \subset [0,1]^2$, and with each $x, y \in V_n$ connected by an edge with probabiltiy $\phi(|y-x|)$ (generalises the geometric E-R graph). Then (by a Mecke-type formula)

$$\mathbb{E}[N_0(G(n,\phi))] \approx n \int \exp\left(-n \int \phi(|y-x|)dy\right) dx.$$

with all integrals being over $[0,1]^2$. Our heuristic suggests for large n that

$$\mathbb{P}[G(n,\phi)) \in \mathcal{M}_1] \approx \exp\left(-n \int \exp\left(-n \int \phi(|y-x|)dy\right) dx\right)$$

and

$$\mathbb{P}[G(n,\phi) \in \mathcal{K}] \approx \exp\left(-n \int \exp\left(-n \int \phi(|y-x|)dy\right) dx\right).$$

A class of connection functions

Given connection function $\phi : \mathbb{R}_+ \to [0,1]$, and given $\eta > 0$, let

$$r_\eta(\phi) = \inf\{t \in \mathbb{R}_+ : \phi(t) \leq \eta\phi(t)\}.$$

Let Φ_η be the class of connection functions ϕ with $r_\eta(\phi) > 0$ and

$$\frac{\phi(t)}{\phi(0)} \leq \frac{3}{\eta} \exp\left(- \left(\frac{t}{r_\eta(\phi)} \right)^\eta \right), \qquad \text{for all } t \geq 0.$$

Given η, Φ_η is a class of connection functions that are uniformly exponentially decaying in the η power of distance measured in their characteristic length-scale $r_\eta(\phi)$. Note $\Phi_\eta \subset \Phi_{\eta'}$ for $\eta > \eta'$.
All step functions of the form $\phi(t) = p\mathbf{1}_{[0,r]}(t)$ are in Φ_1.
Also, given $\beta, \gamma > 0$, there exists $\eta > 0$ such for that all $\rho > 0$ the *Rayleigh fading* connection function $\phi_\rho(t) = \exp(-\beta(t/\rho)^\gamma)$ (with $\rho > 0$) lies in Φ_η.

Limit theorem for soft random geometric graphs

(MP 2013+). Suppose $\eta > 0$. Then as $n \to \infty$,

$$\sup_{\phi \in \Phi_\eta} \left| \mathbb{P}[G(n,\phi) \in \mathcal{M}_1] - \exp\left[-n \int \exp\left(-n \int \phi(y-x) dy \right) dx \right] \right| \to 0$$

(where all integrals are over $[0,1]^2$) and

$$\sup_{\phi \in \Phi_\eta} \left| \mathbb{P}[G(n,\phi) \in \mathcal{K}] - \exp\left[-n \int \exp\left(-n \int \phi(y-x) dy \right) dx \right] \right| \to 0.$$

Thus

$$\sup_{\phi \in \Phi_\eta} |\mathbb{P}[G(n,\phi) \in \mathcal{K}] - \mathbb{P}[G(n,\phi) \in \mathcal{M}_1]| \to 0.$$

Random geometric bipartite ('AB') graphs

Suppose $W_m = \{Y_1, \ldots, Y_m\}$ with Y_i also uniform on $[0,1]^2$.
Let $G_2(r,n,m)$ be the bipartite graph with vertex set $V_n \cup W_m$ (actually Poissonized) and edge set

$$E(G_2(r,n,m)) = \{\{V_i, W_j\} : |V_i - W_j| \leq r\}.$$

It is a geometric graph with edges only between vertices of opposite type.
Let $G_1^2(r,n,m)$ be the graph on V_n with X_i, X_j connected iff they have a common neighbour in $G^2(r,n,m)$, i.e.

$$E(G_1^2(r,n,m)) = \{\{X_i, X_j\} : \exists Y_k \in W_m \text{ with } |X_i - Y_k| \leq r, |X_j - Y_k| \leq r\}$$

Define $G_2^2(r,n,m)$ similarly on vertex set W_m. Then $G_2(r,n,m)$ is connected iff $G_1^2(r,n,m)$ and $G_2^2(r,n,m)$ are both connected.

Thresholds for bipartite graphs

Let $\rho'_{n,m}$ be the connectivity threshold for $G_1^2(\cdot, n, m)$:

$$\rho'_{n,m} := \min\{r : G_1^2(r, n, m) \in \mathcal{K}\}$$

Weak law of large numbers (MP 2013+). Let $\tau > 0$. Then as $n \to \infty$,

$$n\pi(\rho'_{n,\tau n})^2 / \log n \xrightarrow{P} \max(1/\tau, 1/4)$$

(Improves on asymptotic bounds of Iyer and Yogeshwaran 2012).
Expect the easiest way to be disconnected is to have an isolated point.
Easiest way for X_i to be isolated is: to have no Y-points within r (if $\tau \leq 4$), or to have no X-points within $2r$ (if $\tau \geq 4$). E.g. if $n\pi r_n^2 / \log n = 1/\tau$ then the expected number of X_i with no Y-neighbours is

$$n \exp(-n\tau\pi r_n^2) = n \exp(-\log n) = 1$$

Hamiltonian paths

A graph is said to be *Hamiltonian* if there is a travelling salesman tour through the vertices, i.e. a cycle through the vertices using edges of the graph. Let H be the set of hamiltonian graphs with more than two vertices. Clearly $H \subset \mathcal{M}_2$.
Balogh, Bollobas, Krivelevich, Müller and Walters (2011) proved that

$$\lim_{n \to \infty} P[\rho_n(H) = \rho_n(\mathcal{M}_2)] = 1.$$

That is, if we add edges one by one in order of increasing length, the graph becomes hamiltonian just when the minimum degree goes above 1.

Other probability distributions

Now let $V_n = \{X_1, \ldots, X_n\}$ with $X_1, X_2, \ldots$ independent in $\mathbb{R}^d$, with common probability density f, i.e. with $\mathbb{P}[X_i \in A] = \int_A f(x)dx$ (up to now d was 2 and f was uniform on the unit square).
Consider the *normal* with $f(x) = c \exp(-|x|^2)$. Choose r_n so $\mathbb{E}[N_0(G(n, r_n))] \to \alpha > 0$. Then (MP 1998)

$$\mathbb{P}[N_0(G(n, r_n)) = 0] \to e^{-\alpha};$$
$$\mathbb{P}[G(n, r_n) \in \mathcal{K}] \to e^{-\alpha};$$

and hence

$$\sup_{r \geq 0} \mathbb{P}[G(n, r_n) \in \mathcal{M}_1 \setminus \mathcal{K}] \to 0.$$

For $d = 2$ only, Hsing and Rootzen (2005) generalized this result to a larger class of densities, for example of the form $f(x) = c \exp(-c' \|x\|^\beta)$ with $\beta > 1$, with $\| \cdot \|$ any norm with elliptical contours.