Percolation in hard-core systems

Mathew Penrose
(University of Bath)

Stochastic Neworks and Related Topics III
Bedlewo, May 2011

A BRIEF ADVERTISING MESSAGE

There is a Professorial post in probability currently being advertised at Bath. For details follow the link from

http://www.bath.ac.uk/math-sci/

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Particles arrive at the sites in random order.

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Particles arrive at the sites in random order.

Incoming particle at site x, if accepted, causes neighbouring sites to become permanently blocked.

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Particles arrive at the sites in random order.

Incoming particle at site x, if accepted, causes neighbouring sites to become permanently blocked.

Incoming particle at blocked site is rejected.

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Particles arrive at the sites in random order.

Incoming particle at site x, if accepted, causes neighbouring sites to become permanently blocked.

Incoming particle at blocked site is rejected.

End up with random stable set of occupied sites

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Particles arrive at the sites in random order.

Incoming particle at site x, if accepted, causes neighbouring sites to become permanently blocked.

Incoming particle at blocked site is rejected.

End up with random stable set of occupied sites

Weak limit as $n \to \infty$, a random stable set in Z^2.

Random sequential adsorption in Z^2

Initially empty $n \times n$ chessboard

Particles arrive at the sites in random order.

Incoming particle at site x, if accepted, causes neighbouring sites to become permanently blocked.

Incoming particle at blocked site is rejected.

End up with random stable set of occupied sites

Weak limit as $n \to \infty$, a random stable set in Z^2.

Equivalent: iid exp(λ) arrival times at all $x \in Z^2$.

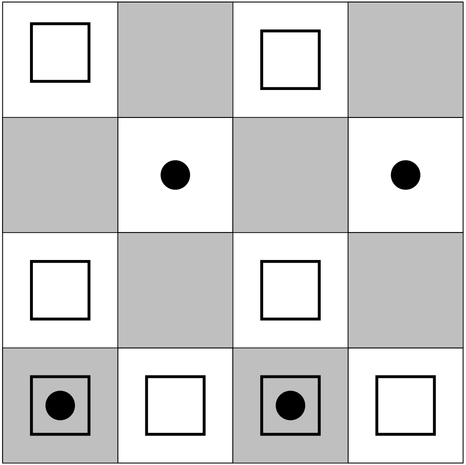


Figure: Squares containing a dot are occupied sites. Squares with an inscribed square in are black sites; the other squares are white sites

Random sequential adsorption (RSA) is a model for irreversible particle deposition onto an initially clean surface.

Random sequential adsorption (RSA) is a model for irreversible particle deposition onto an initially clean surface.

Our lattice version of RSA generates a random maximal stable set in Z^2.

Random sequential adsorption (RSA) is a model for irreversible particle deposition onto an initially clean surface.

Our lattice version of RSA generates a random maximal stable set in Z^2.

We consider *percolation* properties of this set.

Percolation for random sequential adsorption

Given RSA on infinite chessboard, partition sites (squares) into:

Percolation for random sequential adsorption

Given RSA on infinite chessboard, partition sites (squares) into:

Black phase: dark (even) occupied sites and light (odd) blocked sites

White phase: light occupied sites and dark blocked sites

Percolation for random sequential adsorption

Given RSA on infinite chessboard, partition sites (squares) into:

Black phase: dark (even) occupied sites and light (odd) blocked sites

White phase: light occupied sites and dark blocked sites

Does the black phase percolate, i.e. include an infinite component a.s.?

Percolation for random sequential adsorption

Given RSA on infinite chessboard, partition sites (squares) into:

Black phase: dark (even) occupied sites and light (odd) blocked sites

White phase: light occupied sites and dark blocked sites

Does the black phase percolate, i.e. include an infinite component a.s.?

Might expect not (cf indep. site percolation).

Percolation for random sequential adsorption

Given RSA on infinite chessboard, partition sites (squares) into:

Black phase: dark (even) occupied sites and light (odd) blocked sites

White phase: light occupied sites and dark blocked sites

Does the black phase percolate, i.e. include an infinite component a.s.?

Might expect not (cf indep. site percolation).

Considering occupied even sites, maybe not so clear (lattice with diagonals).

Percolation for random sequential adsorption

Given RSA on infinite chessboard, partition sites (squares) into:

Black phase: dark (even) occupied sites and light (odd) blocked sites

White phase: light occupied sites and dark blocked sites

Does the black phase percolate, i.e. include an infinite component a.s.?

Might expect not (cf indep. site percolation).

Considering occupied even sites, maybe not so clear (lattice with diagonals).

In fact, first intuition correct...

Biased RSA

Now assume arrivals are at rate 1 for light sites and rate λ for dark sites.

Biased RSA

Now assume arrivals are at rate 1 for light sites and rate λ for dark sites.

Let λ_c be the infimum of those λ such that the black phase percolates a.s.

Biased RSA

Now assume arrivals are at rate 1 for light sites and rate λ for dark sites.

Let λ_c be the infimum of those λ such that the black phase percolates a.s.

THEOREM (Penrose and Rosoman 2011). we have

$$1 < \lambda_c < 10$$

Biased RSA

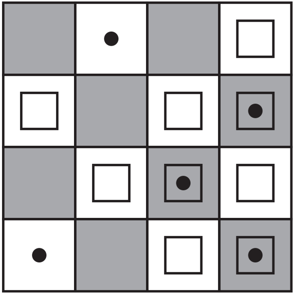
Now assume arrivals are at rate 1 for light sites and rate λ for dark sites.

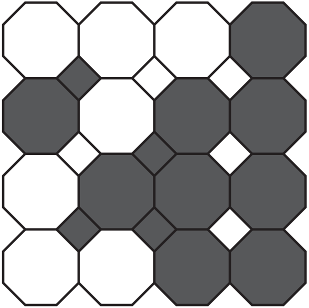
Let λ_c be the infimum of those λ such that the black phase percolates a.s.

THEOREM (Penrose and Rosoman 2011). we have

$$1 < \lambda_c < 10$$

So in particular, when $\lambda = 1$ the black phase does not percolate, a.s. We'll sketch a proof of this weaker result.





Enhancement and duality

Let us add diagonal (diamond) sites. The octagons correspond to the original squares of the chessboard.

Perform RSA as before on the octagons. Independently colour each of the diamonds black with probability p

If $\lambda = 1$ and $p = 1/2$ the model is self-dual and the probabilty of a horizontal black crossing of an $s \times s$ grid is $1/2$.

The Bollobás-Riordan RSW lemma

Let $f(\rho, s)$ be the probabilty of a white crossing of a $\rho s \times s$ rectangle.

If $\liminf_{s\to\infty} f(1, s) > 0$ then $\limsup_{s\to\infty} f(\rho, s) > 0$.

The Bollobás-Riordan RSW lemma

Let $f(\rho, s)$ be the probabilty of a white crossing of a $\rho s \times s$ rectangle.

If $\liminf_{s \to \infty} f(1, s) > 0$ then $\limsup_{s \to \infty} f(\rho, s) > 0$.

Bollobás and Riordan (2006) prove this for a different model but can adapt to the present setting. Need Harris-FKG inequality which is known here (Penrose and Sudbury 2005), and rapidly decaying correlations.

The Bollobás-Riordan RSW lemma

Let $f(\rho, s)$ be the probabilty of a white crossing of a $\rho s \times s$ rectangle.

If $\liminf_{s\to\infty} f(1,s) > 0$ then $\limsup_{s\to\infty} f(\rho,s) > 0$.

Bollobás and Riordan (2006) prove this for a different model but can adapt to the present setting. Need Harris-FKG inequality which is known here (Penrose and Sudbury 2005), and rapidly decaying correlations.

Know $\liminf_{s\to\infty} f(1,s) = 1/2$, so $\limsup_{s\to\infty} f(\rho,s) > 0$.

Proof of non-percolation

Proof of non-percolation

Consider a square annulus A_n round the origin made up of four overlapping $3n \times n$ blocks.

Proof of non-percolation

Consider a square annulus A_n round the origin made up of four overlapping $3n \times n$ blocks.

Using Harris-FKG again, can choose infinite sequence $(n(k), k \geq 1)$ so that the probabilty of a white path round each annulus $A_{n(k)}$ is bounded away from zero.

Proof of non-percolation

Consider a square annulus A_n round the origin made up of four overlapping $3n \times n$ blocks.

Using Harris-FKG again, can choose infinite sequence $(n(k), k \geq 1)$ so that the probabilty of a white path round each annulus $A_{n(k)}$ is bounded away from zero.

Using exponential decay can also ensure successive annuli are almost independent.

Proof of non-percolation

Consider a square annulus A_n round the origin made up of four overlapping $3n \times n$ blocks.

Using Harris-FKG again, can choose infinite sequence $(n(k), k \geq 1)$ so that the probabilty of a white path round each annulus $A_{n(k)}$ is bounded away from zero.

Using exponential decay can also ensure successive annuli are almost independent.

Upshot: can arrange so there is a white circuit in one of the annuli, almost surely.

Proof of non-percolation

Consider a square annulus A_n round the origin made up of four overlapping $3n \times n$ blocks.

Using Harris-FKG again, can choose infinite sequence $(n(k), k \geq 1)$ so that the probabilty of a white path round each annulus $A_{n(k)}$ is bounded away from zero.

Using exponential decay can also ensure successive annuli are almost independent.

Upshot: can arrange so there is a white circuit in one of the annuli, almost surely.

Hence, no infinite black path from the origin, almost surely. This is true for $p = 1/2$ so even more true for $p = 0$.

Idea behind proving $\lambda_c > 1$.

Increasing λ (rate on dark sites) equivalent to decreasing μ (rate on light sites).

Idea behind proving $\lambda_c > 1$.

Increasing λ (rate on dark sites) equivalent to decreasing μ (rate on light sites).

Let $\theta_n(\mu, p)$ be the probability of white crossing of $n \times n$ square, with rate μ on light sites, 1 on dark sites, and enhancement parameter p.

Idea behind proving $\lambda_c > 1$.

Increasing λ (rate on dark sites) equivalent to decreasing μ (rate on light sites).

Let $\theta_n(\mu, p)$ be the probability of white crossing of $n \times n$ square, with rate μ on light sites, 1 on dark sites, and enhancement parameter p.

By self-duality, $\theta_n(\mu = 1, p = 1/2) = 1/2$.

Idea behind proving $\lambda_c > 1$.

Increasing λ (rate on dark sites) equivalent to decreasing μ (rate on light sites).

Let $\theta_n(\mu, p)$ be the probability of white crossing of $n \times n$ square, with rate μ on light sites, 1 on dark sites, and enhancement parameter p.

By self-duality, $\theta_n(\mu = 1, p = 1/2) = 1/2$.

Aim to reduce μ slightly below 1 while increasing p above $1/2$, and still have $\theta_n(\mu, p)$ bounded away from zero. Can then repeat previous argument.

Idea behind proving $\lambda_c > 1$.

Increasing λ (rate on dark sites) equivalent to decreasing μ (rate on light sites).

Let $\theta_n(\mu, p)$ be the probability of white crossing of $n \times n$ square, with rate μ on light sites, 1 on dark sites, and enhancement parameter p.

By self-duality, $\theta_n(\mu = 1, p = 1/2) = 1/2$.

Aim to reduce μ slightly below 1 while increasing p above 1/2, and still have $\theta_n(\mu, p)$ bounded away from zero. Can then repeat previous argument.

Can use Russo's formula and enhancement to show for some fixed C that

$$\frac{\partial \theta_n}{\partial \mu} \leq C \frac{\partial \theta_n}{\partial p}$$

The Lilypond Model: a continuum hard-core system

Suppose $\varphi \subset \mathbb{R}^2$ is finite with at least 2 elements.

Grow disks at unit rate from each point, starting all at once.

Each disk stops growing when it hits another disk.

Let $\rho(x) = \rho(x, \varphi)$ be the resulting radii for $x \in \varphi$.

The resulting system of disks (grains) is called the **Lilypond model** on φ. It is the maximin system satisfying the *hard-core* property

$$\rho(x) + \rho(y) \leq |x - y|, \quad x, y \in \varphi.$$

Given countable $\varphi \subset \mathbb{R}^d$ suppose $\rho := (\rho(x), x \in \varphi)$ is a set of radii.

We say $x, y \in \varphi$ are *grain-neighbours* if $\rho(x) + \rho(y) = |x - y|$.

If also $\rho(y) \leq \rho(x)$, say y is a *smaller grain-neighbour* of x.

We say ρ has the *smaller grain-neighbour property* if every $x \in \varphi$ has a smaller grain-neighbour.

For locally finite $\varphi \subset \mathbb{R}^d$, the lilypond model is the **unique** system $\rho(x), x \in \varphi$ satisfying the hard-core and smaller grain-neighbour properties (Heveling/Last 2006). Set $B_r(x) := \{y : |y - x| \leq r\}$ and

$$Z(\varphi) = \cup_{x \in \varphi} B_{\rho(x,\varphi)}(x).$$

For $x \in \varphi$, let $\mathbb{C}(x, \varphi)$ be the component of $Z(\varphi)$ containing x.

Let Φ be a homogeneous (i.e. stationary) Poisison process in $\mathbb{R}^d$.

It is known that $Z(\Phi)$ does not percolate (i.e. has no infinite component, a.s.). (Häggström and Meester 1996).

Analysis difficult because of complicated dependence; inserting a point into φ may affect several radii by a chain reaction.

THEOREM (Last and Penrose 2010). There exists $\delta > 0$ such that $Z^\delta(\Phi)$ does not percolate, where

$$Z^\delta(\varphi) := \cup_{x \in \varphi} B_{\rho(x,\varphi)+\delta}(x).$$

STABILIZATION (Main tool for proofs)

Let $\varphi^0 = \varphi \cup \{0\}$. We can define $R(\varphi) \in [0, \infty]$ such that:

(i) if $R(\varphi) < \infty$ and $\psi \cap B_{R(\varphi)}(0) = \varphi \cap B_{R(\varphi)}(0)$, then

- $R(\psi) = R(\varphi)$ (stopping time property).
- $\rho(0, \varphi^0) = \rho(0, \psi^0)$. (i.e. $\varphi^0 \cap B_{R(\varphi)}(0)$ determines $\rho(0, \varphi)$).

(ii) There are constants c, C such that $P[R(\Phi) > r] \leq C \exp(-cr^{d/(d+1)})$, $\forall r > 0$.

HOW TO DEFINE $R(\varphi)$ (cf. HM 1996, Daley and Last 2005).

Let $DC(\varphi)$ (the set of *descending chains* in φ) be the set of sequences $(x_0, \ldots, x_n)$ of distinct elements of φ such that $(|x_i - x_{i-1}|, 1 \leq i \leq n)$ is nonincreasing.

For $x \in \varphi$ let $N(x, \varphi) = \min\{|y - x| : y \in \varphi \setminus \{x\}\}$. Set

$$R(\varphi) := \sup\{|x_n| + |x_n - x_{n-1}| : (0, \ldots, x_n) \in DC(\varphi^0), |x_1| \leq 2N(0, \varphi^0)\}.$$

Clearly $\rho(0, \varphi^0) < 2N(0, \varphi^0)$, and $\rho(0, \varphi^0) = \rho(0, \varphi^0 \cap B_{R(\varphi)}(0))$ because:

If x_1 affects 0 directly, then $|x_1| \leq 2N(0, \varphi^0)$.
If x_2 affects x_1 before x_1 affects x_0 then $|x_2 - x_1| \leq |x_1|$, etc.

TAIL BEHAVIOUR OF $R(\Phi)$.

If $R(\Phi)$ is large then either $N(0,\Phi^0)$ is large or for some descending chain from 0, there are a lot of links ...

Let $r > 0$. The probability there exists $(0, x_1, \ldots, x_n) \in DC(\Phi^0)$ with $|x_1| \leq r$, is bounded by the expected number of such n-tuples. This is equal to

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{r > |x_1| > |x_2 - x_1| \cdots > |x_n - x_{n-1}|\} dx_1 \ldots dx_n$$

$$= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{r > |x_1| > |y_2| \cdots > |y_n|\} dx_1 dy_2 \ldots dy_n$$

$$= \frac{(\pi_d r^d)^n}{n!}$$

Tail behavour of $R(\Phi)$ comes from decay of $n!^{-1}$.

Moreover we can define a radius $\tilde{R} = \tilde{R}(\varphi)$ with similar tail behaviour for $\tilde{R}(\Phi)$ and with stopping time property and also:

(i) No influence from inside $B_{\tilde{R}/2}(0)$ on radii outside $B_{\tilde{R}}(0)$, or vice versa.

(ii) No component of $Z(\varphi)$ intersects both with $B_{\tilde{R}/2}(0)$ and with $\mathbb{R}^d \setminus B_{\tilde{R}}(0)$.

IDEA BEHIND DEFINITION OF $\tilde{R}(\varphi)$

Suppose each $x \in \varphi$ each vertex has a *unique* smaller grain-neighbour (Φ has this property a.s. (Daley and Last 2005)).

Make a directed graph on φ with (x, y) an edge iff y a smaller grain neighbour of x. Every vertex will have out-degree 1.

If there is a path in the undirected graph across an annulus $B_{4r}(0) \setminus B_{2r}(0)$, then:

either there is a DC from outside $B_{4r}(0)$ to inside $B_{3r}(0)$

or there is a DC from inside $B_{2r}(0)$ to outside $B_{3r}(0)$.

Take $\tilde{R}(\varphi)$ to be 4 times the smallest r such that neither of these possibilities happens.

Sketch proof that $Z^\delta(\Phi)$ does not percolate, some $\delta > 0$.

Let $K > 0, \delta > 0$ (choose later). Divide $\mathbb{R}^d$ into cubes Q_z of side K, indexed by $z \in \mathbb{Z}^d$. Q_z centred at Kz.

Set $Y_z = 0$ iff $\rho(x, \Phi) \leq K$ and $\tilde{R}(-x + \Phi) \leq K$ for all $x \in \Phi \cap Q_z$ and $\rho(x) + \rho(y) + 2\delta < |x - y|$ for all $x, y \in \cup_{z' : \|z' - z\|_\infty = 1} Q_{z'}$ that are not grain-neighbours. Otherwise $Y_z = 1$.

$(Y_z, z \in \mathbb{Z}^d)$ is finite range dependent site percolation, with $\mathbf{P}[Y_z = 1]$ arbitrarily small by choice of K, δ.

If there is an infinite path in $Z^\delta(\Phi)$, there must be an infinite path in the (Y_z) process.

OTHER RESULTS (Last and Penrose 2010)

(i) There are constants c, C such that

$$\mathbf{P}[\mathrm{Diam}(\mathbb{C}(0, \Phi^0)) \geq r] \leq C \exp(-cr^{d/(d+1)}).$$

(iii) Let Φ_n be a Poisson process of intensity n on $[0, n^{1/d}]^d$. Then $\mathrm{Vol}(Z(\Phi_n))$ satisfies a Central Limit Theorem as $n \to \infty$, and so does the number of components of Φ_n. We also have de-Poissonized CLTs.

- Bollobás, B. and Riordan, O. (2006) The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theory Related Fields* **136** (2006), 417–468.
- D.J. Daley, G. Last (2005). 'Descending chains, lilypond growth ...' *Adv. in Appl. Probab.* **37**, 604–628.
- O. Häggström, R. Meester (1996). 'Nearest neighbor and hard sphere models ...' *Random Struct. Algorithms* **9**, 295–315.
- M. Heveling, G. Last (2006). 'Existence, uniqueness and algorithmic computation of general lilypond systems.' *Random Struct. Algorithms* **29**, 338–350.
- G. Last and M. D. Penrose (2010) Percolation and limit theory for the Poisson lilypond model. ArXiv:1008.0769
- M. D. Penrose and T. Rosoman (2011) Percolation of even sites for random sequential adsorption Arxiv:1102.0532