Percolation and limit theory for the Poisson lilypond model

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THE LILYPOND MODEL.

Suppose $\varphi \subset \mathbf{R}^2$ is finite with at least 2 elements.

Grow disks at unit rate from each point, starting all at once.

Each disk stops growing when it hits another disk.

Let $\rho(x) = \rho(x, \varphi)$ be the resulting radii for $x \in \varphi$.

The resulting system of disks (grains) is called the **Lilypond model** on φ. It is the maximin system satisfying the *hard-core* property

$$\rho(x) + \rho(y) \le |x - y|, \quad x, y \in \varphi.$$

Given countable $\varphi \subset \mathbf{R}^d$ suppose $\rho := (\rho(x), x \in \varphi)$ is a set of radii.

We say $x, y \in \varphi$ are *grain-neighbours* if $\rho(x) + \rho(y) = |x - y|$.

If also $\rho(y) \leq \rho(x)$, say y is a *smaller grain-neighbour* of x.

We say ρ has the *smaller grain-neighbour property* if every $x \in \varphi$ has a smaller grain-neighbour.

For locally finite $\varphi \subset \mathbf{R}^d$, the lilypond model is the **unique** system $\rho(x), x \in \varphi$ satisfying the hard-core and smaller grain-neighbour properties (Heveling/Last 2006). Set $B_r(x) := \{y : |y - x| \leq r\}$ and

$$Z(\varphi) = \cup_{x \in \varphi} B_{\rho(x,\varphi)}(x).$$

For $x \in \varphi$, let $\mathcal{C}(x, \varphi)$ be the component of $Z(\varphi)$ containing x.

Let Φ be a homogeneous (i.e. stationary) Poisison process in $\mathbf{R}^d$. No precise formulae are known for quantities of interest such as

$$\mathbb{P}[\rho(0, \Phi^0) \leq t]; \quad \mathbb{E}[\mathrm{vol}(Z(\Phi) \cap [0,1]^d)],$$

where $\varphi^x := \varphi \cup \{x\}$.

It is known that $Z(\Phi)$ does not percolate (i.e. has no infinite component, a.s.). (Häggström and Meester 1996). But little known about

$$\mathbb{P}[\mathrm{Diam}(\mathcal{C}(0, \Phi^0)) \leq t], \quad \mathbb{P}[\sharp(\mathcal{C}(0, \Phi^0)) \leq t], \quad \mathbb{P}[\mathrm{vol}(\mathcal{C}(0, \Phi^0)) \leq t].$$

Analysis difficult because of complicated dependence; inserting a point into φ may affect several radii by a chain reaction.

NEW RESULTS (Last and Penrose 2010)

(i) There are constants c, C such that

$$\mathbb{P}[\mathrm{Diam}(\mathcal{C}(0, \Phi^0)) \geq r] \leq C \exp(-cr^{d/(d+1)}).$$

(ii) There exists $\delta > 0$ such that $Z^\delta(\Phi)$ does not percolate, where

$$Z^\delta(\varphi) := \cup_{x \in \varphi} B_{\rho(x,\varphi)+\delta}(x).$$

(iii) Let Φ_n be a Poisson process of intensity 1 on $[0, n^{1/d}]^d$. Then $\mathrm{vol}(Z(\Phi_n))$ satisfies a Central Limit Theorem as $n \to \infty$, and so does the number of components of Φ_n. We also have de-Poissonized CLTs.

STABILIZATION (Main tool for proofs). Let $\varphi^0 = \varphi \cup \{0\}$. We can define $R(\varphi) \in [0, \infty]$ such that:

(i) if $R(\varphi) < \infty$ and $\psi \cap B_{R(\varphi)}(0) = \varphi \cap B_{R(\varphi)}(0)$, then

- $R(\psi) = R(\varphi)$ (stopping time property).
- $\rho(0, \varphi^0) = \rho(0, \psi^0)$. (i.e. $\varphi^0 \cap B_{R(\varphi)}(0)$ determines $\rho(0, \varphi)$).

(ii) There are constants c, C such that
$P[R(\Phi) > r] \leq C \exp(-cr^{d/(d+1)})$, $\forall r > 0$.

HOW TO DEFINE $R(\varphi)$ (cf. HM 1996, Daley and Last 2005).

Let $DC(\varphi)$ (the set of *descending chains* in φ) be the set of sequences $(x_0, \ldots, x_n)$ of distinct elements of φ such that $(|x_i - x_{i-1}|, 1 \leq i \leq n)$ is nonincreasing.

For $x \in \varphi$ let $N(x, \varphi) = \min\{|y - x| : y \in \varphi \setminus \{x\}\}$. Set

$$R(\varphi) := \sup\{|x_n| + |x_n - x_{n-1}| :$$

$$(0, \ldots, x_n) \in DC(\varphi^0), |x_1| \leq 2N(0, \varphi^0)\}.$$

Clearly $\rho(0, \varphi^0) < N(0, \varphi^0)$, and $\rho(0, \varphi^0) = \rho(0, \varphi^0 \cap B_{R(\varphi)}(0))$ because:

If x_1 affects 0 directly, then $|x_1| \leq 2N(0, \varphi^0)$.

If x_2 affects x_1 before x_1 affects x_0 then $|x_2 - x_1| \leq |x_1|$, etc.

TAIL BEHAVIOUR OF $R(\Phi)$.

If $R(\Phi)$ is large then either $N(0, \Phi^0)$ is large or for some descending chain from 0, there are a lot of links ...

Let $r > 0$. The probability there exists $(0, x_1, \ldots, x_n) \in DC(\Phi^0)$ with $|x_1| \leq r$, is bounded by the expected number of such n-tuples. This is equal to

$$\int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \mathbf{1}\{r > |x_1| > |x_2 - x_1| > \cdots > |x_n - x_{n-1}|\} dx_1 \ldots dx_n$$

$$= \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \mathbf{1}\{r > |x_1| > |y_2| > \cdots > |y_n|\} dx_1 dy_2 \ldots dy_n$$

$$= \frac{(\pi_d r^d)^n}{n!}$$

Tail behaviour of $R(\Phi)$ comes from decay of $n!^{-1}$.

Using tail behaviour of R, can get a CLT for $\mathrm{vol}(Z(\Phi_n))$ using general results (e.g. in Penrose 2007, Penrose and Yukich 2001).

Idea: $Z(\Phi_n)$ is sum of weakly dependent contributions from different regions of space.

In fact we can define a radius $\tilde{R} = \tilde{R}(\varphi)$ with similar tail behaviour for $\tilde{R}(\Phi)$ and with stopping time property and also:

(i) No influence from inside $B_{\tilde{R}/2}(0)$ on radii outside $B_{\tilde{R}}(0)$, or vice versa. [useful to prove de-Poissonized CLTs]

(ii) No component of $Z(\varphi)$ intersects both with $B_{\tilde{R}/2}(0)$ and with $\mathbf{R}^d \setminus B_{\tilde{R}}(0)$. [useful to prove results concerning components]

IDEA BEHIND DEFINITION OF $\tilde{R}(\varphi)$

Suppose each $x \in \varphi$ each vertex has a *unique* smaller grain-neighbour (Φ has this property a.s. (Daley and Last 2005)).

Make a directed graph on φ with (x, y) an edge iff y a smaller grain neighbour of x. Every vertex will have out-degree 1.

If there is a path in the undirected graph across an annulus $B_{4r}(0) \setminus B_{2r}(0)$, then:

either there is a DC from outside $B_{4r}(0)$ to inside $B_{3r}(0)$

or there is a DC from inside $B_{2r}(0)$ to outside $B_{3r}(0)$.

Take $\tilde{R}(\varphi)$ to be 4 times the smallest r such that neither of these possibilities happens.

Sketch proof that $Z^\delta(\Phi)$ does not percolate, some $\delta > 0$.

Let $K > 0, \delta > 0$ (choose later). Divide $\mathbf{R}^d$ into cubes Q_z of side K, indexed by $z \in \mathbf{Z}^d$.

Set $Y_z = 0$ iff $\rho(x, \Phi) \leq K$ and $\tilde{R}(-x + \Phi) \leq K$ for all $x \in \Phi \cap Q_z$ and $\rho(x) + \rho(y) + 2\delta < |x - y|$ for all $x, y \in \cup_{z':\|z'-z\|_\infty=1} Q_{z'}$ that are not grain-neighbours.

Otherwise $Y_z = 1$.

$(Y_z, z \in \mathbf{Z}^d)$ is finite range dependent site percolation, with $\mathbb{P}[Y_z = 1]$ arbitrarily small by choice of K, δ.

If there is an infinite path in $Z^\delta(\Phi)$, there must be an infinite path in the (Y_z) process.

References

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