Connectivity of Some Random Graphs

Mathew Penrose

Bath Networks Seminar
March 2012

Overview

Let K be the set of all connected finite graphs with at least two vertices. Let M_1 be the set of all graphs G with minimum degree at least 1, i.e. with $N_0(G) = 0$ where $N_0(G)$ is the number of isolated vertices. Clearly

$$K \subset M_1.$$

Our main message is that for certain types of large *random* graphs G, the probability

$$\mathbb{P}[G \in \mathcal{M}_1 \setminus K] \quad \text{is small.}$$

This implies an *equivalence result* $\mathbb{P}[G \in K] \approx \mathbb{P}[G \in M_1]$, and the latter probability is typically easier to compute, at least approximately.

The Erdos-Renyi random graph

Let $n \in \mathbb{N}$ and $p \in (0,1)$. The random graph $G(n,p)$ has n vertices, each of the $n(n-1)/2$ possible edges is present with probability p. Recall that an $\mathbb{N}$-valued random variable X has *expected value* $\mathbb{E}[X]$ defined by

$$\mathbb{E}[X] = \sum_{n \geq 1} n\mathbb{P}[X = n].$$

It is not hard to see that

$$\mathbb{E}[N_0(G(n,p)] = n(1-p)^{n-1}$$

and if we choose $p = p_n$ so that $n(1-p_n)^{n-1} \to \beta$ as $n \to \infty$, it turns out that $N_0(G(n,p)$ is approximately *Poisson* for n large, so

$$\mathbb{P}[G(n,p) \in M_1] = \mathbb{P}[N_0(G(n,p)) = 0] \to e^{-\beta}.$$

(if X is Poisson distributed with $\mathbb{E}[X] = \mu$ then $\mathbb{P}[X = 0] = e^{-\mu}$.)

Equivalence result for $G(n, p)$

Erdos and Renyi (1959) proved that if we choose p_n so $\mathbb{E}[N_0(G(n, p_n))] \to \beta$ then

$$\lim_{n\to\infty} P[G(n, p_n) \in M_1 \setminus K] = 0.$$

Then it is easy to deduce that

$$\lim_{n\to\infty} \left(\sup_{0\leq p\leq 1} P[G(n, p) \in M_1 \setminus K] \right) = 0.$$

The random geometric (Gilbert) graph $G(n, r)$

Let $r > 0, n \in \mathbb{N}$. The graph $G(n, r)$ has vertex set $V_n = \{X_1, \ldots, X_n\}$ with X_i independently uniformly distributed over the unit square $[0, 1]^2$. The edges consist of those $\{X_i, X_j\}$ such that $|X_i - X_j| \leq r$, where $|\cdot|$ is the Euclidean norm. See e.g. 'Random Geometric Graphs' (OUP 2003).
Sometimes more convenient to consider $G(V_{N_n}, r)$ where N_n is Poisson (n), but results are similar - we'll ignore this distinction.
Modulo boundary effects, for r small we have

$$\mathbb{E}[N_0(G(n, r))] = n(1 - \pi r^2)^{n-1} \sim n \exp(-\pi n r^2)$$

If we choose r_n so $n \exp(-\pi n r_n^2) \to \beta \in \mathbb{R}$ then we have another Poisson approximation result (Dette and Henze 1989):

$$\lim_{n \to \infty} P[N_0(G(n, r_n)) = 0] = \exp(-\beta)$$

Equivalence result for $G(n, r)$

If we choose r_n so $\mathbb{E}[N_0(G(n, r_n))] \to \beta$ then

$$\lim_{n\to\infty} P[G(n, r_n) \in M_1 \setminus K] = 0.$$

It is then easy to deduce that

$$\lim_{n\to\infty} \left(\sup_{r \geq 0} P[G(n, r) \in M_1 \setminus K] \right) = 0.$$

The proof of these results is entirely different from in the case of $G(n, p)$. Essentially they are due to MP (1997) although often credited to Gupta and Kumar (1998).

Multiple connectivity, and thresholds

Let $k \in \mathbb{N}$. We say a graph is k-connected if for any two vertices x, y there exist k disjoint paths from x to y. Let K_k be the set of k-connected graphs with at least two vertices. Clearly $K_k \subset M_k$, the set of graphs of minimum degree at least k. MP (1999) showed the equivalence

$$\lim_{n\to\infty} \sup_{r>0} \mathbb{P}[G(n,r) \in M_k \setminus K_k] = 0.$$

Given a realization of $V_n = \{X_1, \ldots, X_n\}$, the *threshold radius* for k-connectivity, respectively the threshold radius for minimum degree at least k, are defined by

$$\rho_n(K_k) = \inf\{r : G(n,r) \in K_k\};$$
$$\rho_n(M_k) = \inf\{r : G(n,r) \in M_k\}.$$

These are random variables, determined by the configuration V_n.

Strong equivalence

Let $k \in \mathbb{N}$. Since $K_k \subset M_k$ we have that $\rho_n(K_k) \geq \rho_n(M_k)$. A strong version of the preceding equivalence holds (MP 1999), namely

$$\lim_{n\to\infty} P[\rho_n(K_k) = \rho_n(M_k)] = 1.$$

That is, if we add the edges amongst V_n one by in order of increasing length, the graph becomes k-connected at the same time as the minimum degree goes above $k-1$. In particular

$$\lim_{n\to\infty} P[\rho_n(K) = \rho_n(M_1)] = 1.$$

It is not known if there is a random finite N such that

$$P[\rho_n(K) = \rho_n(M_1), \quad \forall n \geq N] = 1.$$

Hamiltonian paths

A graph is said to be *Hamiltonian* if there is a travelling salesman tour through the vertices, i.e. a cycle through the vertices using edges of the graph. Let H be the set of hamiltonian graphs with more than two vertices. Clearly $H \subset M_2$.
Balogh, Bollobas, Krivelevich, Müller and Walters (2011) proved that

$$\lim_{n\to\infty} P[\rho_n(H) = \rho_n(M_2)] = 1.$$

That is, if we add edges one by one in order of increasing length, the graph becomes hamiltonian just when the minimum degree goes above 1.

Other probability distributions

Now let $V_n = \{X_1, \ldots, X_n\}$ with Suppose $X_1, X_2, \ldots$ are independent in $\mathbb{R}^d$, with common probability density f, i.e. with $\mathbb{P}[X_i \in A] = \int_A f(x)dx$ (up to now d was 2 and f was uniform on the unit square).
Consider the *normal* with $f(x) = c\exp(-|x|^2)$. Choose r_n so $\mathbb{E}[N_0(G(n, r_n))] \to \alpha > 0$. Then (MP 1998)

$$\mathbb{P}[N_0(G(n, r_n)) = 0] \to e^{-\alpha};$$
$$\mathbb{P}[G(n, r_n) \in K] \to e^{-\alpha};$$

and hence

$$\sup_{r \geq 0} \mathbb{P}[G(n, r_n) \in M_1 \setminus K]$$

For $d = 2$ only, Hsing and Rootzen (2005) generalized this result to a larger class of densities, for example of the form $f(x) = c\exp(-c'\|x\|^\beta)$ with $\beta > 1$, with $\|\cdot\|$ any norm with elliptical contours.

The choice of r_n

Given f and d, and given $\beta \in \mathbb{R}$, let $r_n(\beta, f)$ be chosen so that $\mathbb{E}[N_0(G(n, r_n)) \to \beta$. This is sensitive to the choice of f, e.g.

• For f uniform in $[0,1]^2$,

$$r_n = \sqrt{\frac{\log n - \log \beta + o(1)}{n\pi}}$$

• This carries through to the d-torus, but in the d-cube there are extra boundary effects for $d \geq 3$.

• For f normal, setting $\log_2 n = \log(\log n)$ and $\log_3 n = \log(\log_2 n)$,

$$r_n = \frac{(d-1)\log_2 n - ((d-1)/2)\log_3 n - \log \beta + o(1)}{\sqrt{2(\log n + ((d/2) - 1)\log_2 n - \log \Gamma(d/2))}}$$

• $f(x) = c \exp(-c'|x|^\beta)$ has $r_n \to \infty$ for $\beta \leq 1$

The Geometric Erdos-Renyi graph

Given $n \in \mathbb{N}$, $p \in (0,1]$, and $r > 0$, Let $G(n,r,p)$ be the graph with vertex set V_n (n independent uniform in $[0,1]^2$) and with vertices x, y connected with probability p whenever $|x-y| \leq r$ (and not connected whenever $|x-y| > r$). This is the intersection of the graphs $G(n,p)$ and $G(n,r)$. If $p \in (0,1)$ is fixed and $n(1-\pi p r_n^2)^n \to \beta$, then (MP 2012+)

$$\lim_{n\to\infty} \mathbb{P}[G(n,r_n,p) \in K] = \lim_{n\to\infty} \mathbb{P}[G(n,r_n,p) \in M_1] = \exp(-\beta)$$

and hence $\lim_{n\to\infty} \sup_{r>0} \mathbb{P}[G(n,r,p) \in M_1 \setminus K] = 0$. Still unknown whether

$$\lim_{n\to\infty} \sup_{r>0, p\in(0,1]} \mathbb{P}[G(n,r,p) \in M_1 \setminus K] = 0.$$

That is, if p_n varies and r_n chosen so $\mathbb{E}[N_1(G(n,r_n,p_n)) \to \beta]$, does $\mathbb{P}[G(n,r_n,p_n) \in M_1 \setminus K] \to 0$? OK for p_n bounded away from zero, in fact for $p_n \gg 1/\log n$.

The Eschenauer-Gligor random key scheme

random geometric graphs are motivated by wireless communication networks. The EG random key scheme is a cryptographic device to make these networks more secure, and goes as follows. Let $k, \ell \in \mathbb{N}$ with $k < \ell/2$. Each vertex $x \in V_n$ is assigned a set $\mathcal{K}(x) \subset \{1, 2, \ldots, \ell\}$, chosen uniformly at random from the $\binom{\ell}{k}$ subsets with k ellements. Then vertices $x, y \in V_n$ are connected if
(i) $|x - y| \leq r$
(ii) $\mathcal{K}(x) \cap \mathcal{K}(y) \neq \emptyset$.
This gives a dependent $G(n, r, p)$ with $p = 1 - \binom{\ell-k}{k} / \binom{\ell}{k}$ If the parameters k, ℓ are fixed then we have a dependent the preceding result carries through. However it might be more realistic to consider k_n, ℓ_n with resulting $p_n \to 0$.