[1 sta. loc fin: simple]

69

Prop 7.8 Let $g \in \mathcal{B}(\underline{N}_s)$ bounded (e.g. $g = 1_A$, $A \in \mathcal{N}$). Then putting $\xi = \tau(0, \eta)$

$$\lim_{n\to\infty} E[g(T_{-\xi}\eta) \mid \eta(B_n) \geq 1] = E[g(\eta^0)] \quad (A)$$

if $x \mapsto g(T_x \mu)$ is cts. at 0 $\forall \mu \in \underline{N}_s$ then

$$\lim_{n\to\infty} E[g(\eta) \mid \eta(B_n) \geq 1] = E[g(\eta^0)] \quad (B)$$

Proof Assume $g \geq 0$. Let $B_n' = \{x \in \mathbb{R}^d : |x| \leq 3r_n\}$, s.t.

$B_n'' = \{ \cdots\cdots\cdots 4r_n\}$

$f(x,\mu) = 1_{\{x \in B_n\}} \times g(\mu) \times 1_{\mu(B_n')=1}$ $x \in \mathbb{R}^d$, $\mu \in \underline{N}$.

By Refined Campbell Thm (Thm 7.5),

$$\frac{1}{\gamma \lambda_d(B_n)} E \int f(x, T_{-x}\eta)\, \eta(dx) = \gamma \int E f(x, \eta^0)\, dx / \gamma \lambda_d(B_n)$$

$$RHS = E[g(\eta^0)\, 1_{\eta^0(B_n')=1}]$$

$$\longrightarrow E[g(\eta^0)] \quad \text{by MON}$$

$$LHS = \frac{1}{\gamma \lambda_d(B_n)} E \overbrace{\int_{B_n} g(T_{-x}\eta)\, 1_{\{T_{-x}\eta(B_n')=1\}}\, \eta(dx)}^{V_n}$$

Let $A_n = \{\exists z : \eta|_{B_n} = \delta_z \text{ and } T_{-z}\eta(B_n') = 1\}$

$\exists$ const K with

$V_n \le K\,\eta(B_n) \le K\,\eta(B_n'')$,

If $\eta(B_n) = 0$ then $V_n = 0$

Also $\{\eta(B_n) > 0\} \cap A_n^c \subset \{\eta(B_n'') \ge 2\}$ ($)

$$E[V_n \mathbb{1}_{A_n^c}] = E[K\,\eta(B_n'')\,\mathbb{1}_{\eta(B_n'') \ge 2}]$$

$$= o(r_n^d) \quad \text{by Prop 7.7}$$

If A_n occurs then $Z = \xi$ (N.N. of 0 in supp η)
and $\eta(A_n) = 1$

So $V_n = g(T_{-Z}\eta)$

So "LHS" $= \frac{1}{\gamma \lambda_d(B_n)} E[V_n \mathbb{1}_{A_n}] + o(1)$

$$= \frac{1}{\gamma \lambda_d(A_n)} E[g(T_{-\xi}\eta)\mathbb{1}_{A_n}] + o(1)$$

$$= \frac{1}{\gamma \lambda_d(A_n)} E[g(T_{-\xi}\eta)\mathbb{1}_{\eta(B_n)\ge 1} \quad \mathbb{1}_{A_n}] + o(1)$$

$$= \frac{1}{\gamma \lambda_d(A_n)} E[g(T_{-\xi}\eta)\mathbb{1}_{\eta(A_n)\ge 1}] + o(1) \quad \text{using } (\$)$$

$$= E[g(T_{-\xi}\eta) \mid \eta(B_n)\ge 1] + o(1)$$

$P[\eta(B_n) \ge 1](1 + o(1))$ by Prop 7.7

So (A) holds. For (B), put $f(x,\mu) = 1_{\{x \in B_n\}} g(\mu)$.

By alternative refined campbell:

$$\frac{1}{\gamma \lambda_d(B_n)} E \int f(x,\eta)\, \eta(dx) = \frac{1}{\lambda_d(B_n)} \int E f(x, T_x \eta^0)\, dx$$

$$RHS = \frac{1}{|B_n|} \int_{B_n} E\, g(T_x \eta^0)\, dx$$

By extra assumption on g, & DOM:

$$E\, g(T_x \eta^0) \to E\, g(\eta^0)$$

Hence $RHS \to E\, g(\eta^0)$

$$\text{"}LHS\text{"} = \frac{1}{\gamma |B_n|} E[g(\eta) \cdot \eta(B_n)]$$

$$= \underbrace{\frac{1}{\gamma |B_n|} E[g(\eta) 1_{\{\eta(B_n) \geq 1\}}]}_{①} + \underbrace{\frac{1}{\gamma |B_n|} E[g(\eta)(\eta(B_n) - 1_{\eta(B_n) = 1})]}_{②}$$

$$① = \frac{E[g(\eta) 1_{\{\eta(B_n) \geq 1\}}]}{P[\eta(B_n) \geq 1]} \times \frac{P[\eta(B_n) \geq 1]}{\gamma |B_n|}$$

$$= E[g(\eta) \mid \eta(B_n) \geq 1] \times (1 + o(1))$$

set $K = \sup g(y)$:

② $\leq \underbrace{K}_{\gamma(B_n)} E[\eta(B_n) \mathbb{1}_{\eta(B_n) \geq 2}]$

$\to 0$ see proof of Prop 7.7

& (B) follows

<u>Remark</u> For $\mu, \mu_1, \mu_2, \dots$ in $\underline{N}_S(\mathbb{R}^d)$ we say $\mu_n \xrightarrow{v} \mu$ ($\mu_n \to \mu$ in the vague topology) if $\mu_n(f) \to \mu(f)$ $\forall f \in C_K^+(\mathbb{R}^d)$ where $C_K^+(\mathbb{R}^d)$ is the space of continuous nonnegative functions with compact support.

The vague topology is known to be metrisable making $\underline{N}_S(\mathbb{R}^d)$ into a CSMS.

Now suppose $q : \underline{N}_S(\mathbb{R}^d) \to \mathbb{R}$ is bounded and continuous (w.r.t. vague top.)

Then for $\mu \in \underline{N}_{s}^{K}(\mathbb{R}^d)$, can write

$$\mu = \sum_{i=1}^{K} \delta_{x_i} \quad \text{with } x_i \to \infty \text{ if } K=\infty$$

and for $f \in C_K^+(\mathbb{R}^d)$:

$$\mu(f) = \sum_{i=1}^{K} f(x_i) \qquad \text{(all but finitely many terms zero)}$$

$$T_x\mu(f) = \sum_{i=1}^{K} f(x_i + x) \qquad \text{''}$$

$$T_x\mu(f) \to \mu(f)$$

so $T_x\mu \xrightarrow{v} \mu$

so $g(T_x\mu) \to g(\mu)$

so [for η sta. loc finite simple] by (B) of Prop 7.8, $E[g(\eta) \mid \eta(B_n) \geq 1] \to E[g(\eta^0)]$

In other words.

$$\mathcal{L}(\eta \mid \eta(B_n) \geq 1) \to \mathcal{L}(\eta^0)$$

in the sense of weak convergence of $\underline{N}_S$- valued RVs, with $\underline{N}_S$ given the vague top.

[X_n RVs in metric space,

$X_n \xrightarrow{D} X$ (or $\mathbb{P}_{X_n} \xrightarrow{\text{weak}} P_X$)

means $E\, g(X_n) \to E\, g(X)$

$\forall$ bdd cts. g.]

§8 Poisson Processes on $\mathbb{R}$

Suppose $X = \mathbb{R}^+ = [0,\infty)$. Given loc. finite pt pr η we can write

$$T_n = \inf \{t : \eta(0,t] \geq n\}$$

(with ($\inf \emptyset = +\infty$). Then T_n is a RV, the n-th arrival time.

A PPP on $\mathbb{R}^+$ with intensity $\gamma \lambda d$ ($\gamma \in (0,\infty)$) is called a homogenous Poisson PT. (HPP) with intensity γ.

(Interval Theorem)

Thm 8.1 Suppose η is a loc. finite pt. pr. in $\mathbb{R}^+$, with T_n as above, let $\gamma > 0$. TFAE:

(1) η is a HPP with intensity γ

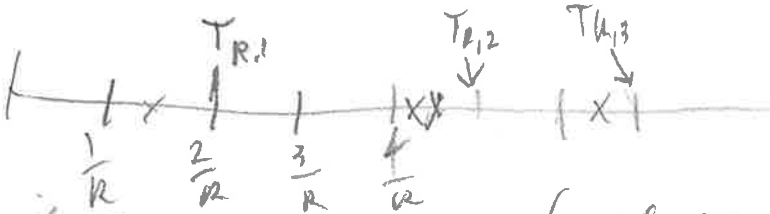
(2) $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are indep $\exp(\gamma)$ RVs.

<u>proof</u> Assume (1).

For $k, n \in \mathbb{N}$ let $I_{k,n} = \left(\frac{n-1}{k}, \frac{n}{k}\right]$

Say $I_{k,n}$ is occupied if $\eta(I_{k,n}] > 0$.

Let $T_{k,n}$ = right-endpoint of n-th occupied interval in the sequence $(I_{k,1}, I_{k,2}, \cdots)$



Put $S_n = T_n - T_{n-1}$ & $S_{k,n} = T_{k,n} - T_{k,n-1}$ (with $T_0 = T_{k,0} = 0$)

Then $T_{k,n} \to T_n$ a.s. so $S_{k,n} \to S_n$ a.s.

But $(S_{k,n})_{n \geq 1}$ are iid with

$$P[S_{k,1} > x] = P[\eta((0,x]) = 0] = e^{-\gamma x}, \text{ if } kx \in \mathbb{N}$$

So $P[S_{k,1} > \cancel{x}] \to e^{-\gamma x}$ as $k \to \infty$ $\forall x \in \mathbb{R}^+$.

$$S_{k,n} \xrightarrow{a.s.} T_n$$

$$S_{k,n} \xrightarrow{\mathcal{D}} \exp(\gamma)$$

so T_n is $\exp(\gamma)$.

$S_{k,1}, S_{k,2}, \cdots$ indep

$S_{k,n} \xrightarrow{a.s.} T_n$

So $T_1, T_2, \cdots$ indep.

(2) $\Longrightarrow$ (1). Assume (2).

Let η' be a HPP with intensity γ.

$$T_n' = \inf\{s : \eta'(s) \geq n\}$$

by first part of the proof

$$(T_n)_{n\geq 1} \overset{d}{=} (T_n')_{n\geq 1}$$

and

$$\eta(B) = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \in B\}}$$ measurable fn of B

So $\eta(B_1), \cdots \eta(B_k) \overset{d}{=} \eta'(B_1), \cdots \eta'(B_k)$

$\forall B_1, \cdots B_k$.

9. Miscellaneous

Thm 9.1 Suppose X is a CSMS and η is a pt pr with $P[\eta \in \underset{\sim}{N}_s] = 1$ and η has indep increments, ie $\forall B_1, B_2, \dots B_k \in \mathcal{X}$ disjoint

$\eta(B_1), \dots, \eta(B_k)$ are indp.

Suppose also $\eta(\{x\}) = 0$ a.s. $\forall x \in X$. Then η is a PP.

Proof wlog $\eta(X) < \infty$ a.s.

(since $\eta(B) < \infty$ a.s. $\forall$ balls B, see Prop 3.2)

also wlog $X = I = [0,1]$ (see Thm 6.1)

We prove $P[\eta(I) = 0] > 0$. Given $t \in I$,

$$\lim_{n \to \infty} P[\eta((t - \tfrac{1}{n}, t + \tfrac{1}{n})) = 0] = P[\eta(\{t\}) = 0] = 1$$

pick $n(t)$ so $P[\eta(I_t) = 0] > 0$ where $I_t = (t - \frac{1}{n}, t + \frac{1}{n})$

$\{I_t : t \in I\}$ is an open cover for I

which is compact, so $\exists\ t_1, \dots t_k$:

$$I \subset \bigcup_{j=1}^{k} I_{t_j}$$

Let $A_1 = I_{t_1}$, $A_2 = I_{t_2} \setminus I_{t_1}$

$$A_{j+1} = I_{t_{j+1}} \setminus \bigcup_{i=1}^{j} I_{t_i} \qquad j \geq 2$$

$I \subset \bigcup_{j=1}^{k} A_j$ and A_j disjoint, $A_j \subseteq I_{t_j}$

$$P[\eta(I)=0] \leq P[\bigcap_{j=1}^{k} \{\eta(A_j)=0\}]$$

$$= \prod_{j=1}^{k} P[\eta(A_j)=0]$$

$$> 0$$

Now put $\lambda(A) = -\log P[\eta(A) = 0]$, $A \in \mathcal{B}(I)$.

For $A_1, A_2, \cdots$ disjoint

$$\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = -\log P\left[\bigcap_{i=1}^{\infty} \{\eta(A_i) = 0\}\right]$$

$$= \lim_{n\to\infty}\left(-\log P\left[\bigcap_{i=1}^{n} \eta(A_i) = 0\right]\right)$$

$$= \lim_{n\to\infty} -\log\left(\prod_{i=1}^{n} P\{\eta(A_i) = 0\}\right) \text{(independence)}$$

$$= \sum_{i=1}^{\infty} \lambda(A_i)$$

So λ is a measure, $\lambda(I) < \infty$ and

$$P[\eta(A) = 0] = e^{-\lambda(A)} \qquad A \in \mathcal{B}(\mathcal{B}).$$

By Renyi's thm

η is a PPP with intensity λ.